Computational Cognitive Science

Lecture 5: Parameters and Probabilities 2

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Readings

- Chapter 6 of F&L

MLE/MAP Recap

Last time, we discussed MLE and MAP estimates:

- MLE: Choose the $\boldsymbol{\theta}$ that makes $\mathbf{y}$ most probable, ignoring $p(\boldsymbol{\theta})$.
- MAP: Choose the $\boldsymbol{\theta}$ that is most probable given $\mathbf{y}$.

MAP with non-uniform priors can improve estimates and reduce overfitting.

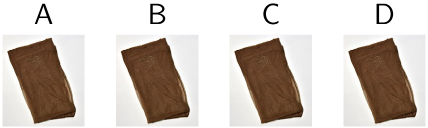
In general, parameters are continuous, so the MAP maximizes the *density* — the probability that the parameters take those values is still infintesimal.

Today

- Estimating parameters in a simple discrete-choice experiment
- Compare MLE, MAP, and Bayesian methods
- Brief introduction to conjugate priors

Left-right bias

In 1977, Nisbett and Wilson reported a study where people has been asked to choose between four identical pairs of stockings: A, B, C, D from left to right[1].



This is similar to the coin example from C6 of F&L, but it involves more than two outcomes and is about human decisions.

[1] Nisbett, R. E., & Wilson, T. D. (1977). Telling more than we can know: Verbal reports on mental processes. *Psychological Review*, 84(3), 231.

Left-right bias

Suppose we observe 8 choices.

Choice	A	B	C	D
Number	0	2	2	4

We want to know how biased people are in general and predict the judgments of the remaining 44 participants.

We can capture this with a multinomial distribution:

$$P(\mathbf{y}|\boldsymbol{\theta}) = \frac{(\sum_i y_i)!}{\prod_i y_i!} \prod_i \theta_i^{y_i}$$

where y_i is the count of choices in the i^{th} category and $\sum_i \theta_i = 1$.

What are our options for estimating $\boldsymbol{\theta}$?

Left-right bias

1. MLE
2. MAP
3. Bayesian approaches

1. MLE: $\arg\max_{\boldsymbol{\theta}} L(\boldsymbol{\theta}|\mathbf{y})$

The multinomial's parameters are choice probabilities, and one can show that the MLE parameters are just the proportions:

$$\theta_i = \frac{y_i}{\sum_k y_k}$$

Choice	A	B	C	D
Obs. (n=8)	0	2	2	4
θ_{MLE}	0	.25	.25	.5

This maximizes the probability of the data in retrospect, but it's not ideal for predictions.

For example, someone will probably, eventually, choose option A.

2. MAP estimate: $\arg\max_{\boldsymbol{\theta}} L(\boldsymbol{\theta}|\mathbf{y})p(\boldsymbol{\theta})$

If we know or believe something about choices in this setting, we should probably use it.

- We might expect that any bias won't be extreme; a few people will probably choose every option
- Like a coin where we expect to be close to fair

How do we express this?

Priors

In choosing priors, we ideally want a distribution that:

- has support for all remotely possible values, i.e., assigns non-zero probability to them
- is easy to interpret and communicate
- allows efficient computation of a posterior distribution

Dirichlet distribution

There are many options, but here we might use a *Dirichlet distribution* with *hyperparameters* $\boldsymbol{\alpha}$:

$$p(\boldsymbol{\theta}|\boldsymbol{\alpha}) = \frac{1}{B(\boldsymbol{\alpha})} \prod_i \theta_i^{\alpha_i - 1}$$

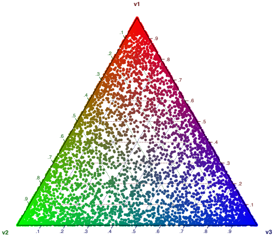
- Can capture intiutions about differences in proportions and in concentration
  - $\boldsymbol{\alpha}$: "Concentration parameters"
  - $\alpha_i > 0$, one per θ
  - "virtual observations"
  - Can translate beliefs about $P(c_0 < \theta_i < c_1)$ into hyperparameters
- Familiar to many cognitive scientists

Dirichlet distribution

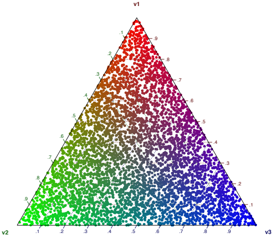
$$p(\boldsymbol{\theta}|\boldsymbol{\alpha}) = \frac{1}{B(\boldsymbol{\alpha})} \prod_i \theta_i^{\alpha_i - 1}$$

The beta distribution (F&L C6) is 2-group Dirichlet distribution.

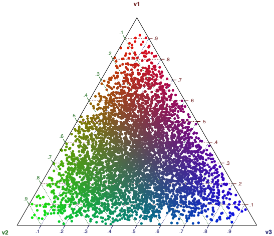
$\boldsymbol{\alpha} = [.5\ .5\ .5]$



$\boldsymbol{\alpha} = [1\ 1\ 1]$



$\boldsymbol{\alpha} = [2\ 2\ 2]$



Also, it is a *conjugate prior* for the multinomial distribution.

Conjugate priors

When the posterior probability function and the prior have the same form, they're *conjugate*.

If we can find a reasonable conjugate prior for our likelihood function, life is easier:

- Simplifies computation
- Makes interpretation of the posterior easier

Conjugate priors

Some commonly-used likelihood/conjugate prior pairs:

Likelihood	Conjugate prior
Bernoulli	beta
binomial	beta
categorical	Dirichlet
multinomial	Dirichlet
normal	normal

Dirichlet-multinomial

Our prior:

$$p(\boldsymbol{\theta}|\boldsymbol{\alpha}) = \frac{1}{B(\boldsymbol{\alpha})} \prod_i \theta_i^{\alpha_i - 1} \propto \prod_i \theta_i^{\alpha_i - 1}$$

Our likelihood:

$$P(\mathbf{y}|\boldsymbol{\theta}) = \frac{(\sum_i y_i)!}{\prod_i y_i!} \prod_i \theta_i^{y_i} \propto \prod_i \theta_i^{y_i}$$

Our posterior:

$$p(\boldsymbol{\theta}|\mathbf{y}) \propto \prod_i \theta_i^{\alpha_i - 1} \prod_i \theta_i^{y_i} = \prod_i \theta_i^{\alpha_i - 1 + y_i}$$

This is an un-normalized Dirichlet distribution; divide by $B(\boldsymbol{\alpha} + \mathbf{y})$ and we have a Dirichlet. See text for more detail in a 2-choice setting.

Dirichlet-multinomial

Our prior is Dir$(\alpha_1, ..., \alpha_K)$ and our posterior is Dir$(\alpha_1 + y_1, ..., \alpha_K + y_K)$.

- We can think of $\boldsymbol{\alpha}$ as pseudo-observations.
- If we believe a bias for each group is equally likely, $\alpha_i = \alpha$.
- As α increases, the Dirichlet distribution increasingly favors an equal distribution over choices.
- For $\alpha = 1$, all valid parameter combinations are equally likely.
- $\alpha = 1/K$ is a Jeffreys prior; see the text.

Left-right bias

If we think extreme biases are unlikely, we can use $\alpha = 2.0$.[2]

The mode of a Dirichlet distribution is $\theta_i = \frac{\alpha_i - 1}{\sum_k \alpha_k - K}$.

Choice	A	B	C	D
Obs. (n=8)	0	2	2	4
θ_{MLE}	0	.25	.25	.5
θ_{MAP}	.08	.25	.25	.42

[2] Under this choice, each parameter's marginal probability of being between .1 and .4 is about 70 percent.

Left-right bias

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Choice	A	B	C	D
Obs. (n=8)	0	2	2	4
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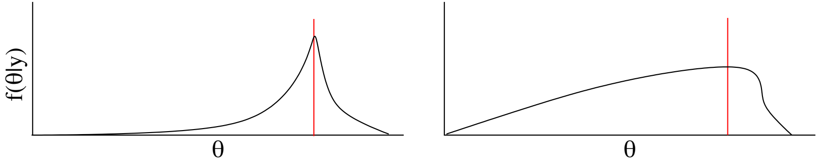
θ_{MAP}	.08	.25	.25	.42
Data (n=52)	.12	.17	.31	.40

3. Bayesian approaches

The MAP estimate doesn't account for uncertainty or the shape of $p(\boldsymbol{\theta}|\mathbf{y})$.

- If our prior is uniform, MAP = MLE; we're back to estimating zero-probabilities.

Compare two densities, where θ is the bias of a coin toward heads:



- The mode of the posterior is the same for both.
- Which are we more confident of?
- Which coin do we think is more likely to come up heads?

3. Bayesian approaches

If we have a posterior distribution, we can ask:

- What is the expected value of α_i?[3]

$$E[\theta_i|\mathbf{y}] = \int_{\theta_i} \theta_i f(\theta_i|\mathbf{y}) d\theta_i$$

- What is the probability that the next choice/toss will be in category i ($x_{K+1} = i$)?

$$P(x_{K+1} = i|\mathbf{y}) = \int_{\theta_i} P(x_{K+1} = i|\theta_i) f(\theta_i|\mathbf{y}) d\theta_i$$

Because θ_i *is* $P(x_{K+1} = i|\theta_i)$, these are the same (here).

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[3] Notice that we're writing down $f(\theta_i|\mathbf{y})$ directly – we get this distribution for free; another nice property of the Dirichlet distribution.

3. Bayesian approaches

For a Dirichlet-multinomial,

$$P(x_{K+1} = i|\mathbf{y}, \boldsymbol{\alpha}) = \frac{\alpha_i + y_i}{\sum_j (\alpha_j + y_j)}$$

Choice	A	B	C	D
Obs. (n=8)	0	2	2	4
θ_{MLE}	0	.25	.25	.5
θ_{MAP}	.08	.25	.25	.42
$P(x_{K+1} = i)$	.12	.25	.25	.38
Data (n=52)	.12	.17	.31	.40

3. Bayesian approaches

We can also answer other questions, e.g.,

- How likely is it that people are choosing option D more than 25 percent of the time?
  - $P(\theta_4 > .25|\mathbf{y})$
- How likely is it that people are choosing uniformly (null hyp)?
  - For all i, $P(\theta_i = .25 \pm \epsilon|\mathbf{y})$
- What is the standard deviation of θ_i?
- What is the probability that $\theta_1 + \theta_2 > \theta_3 + \theta_4$?

3. Bayesian approaches

To summarize, some advantages of Bayesian approaches over MLE:

- Sensible priors and averaging both help us avoid overfitting
  - This allows more complex models, including cases where MLEs aren't unique
- Can answer diverse questions, e.g., support *for* null hypotheses
- Naturally lead to hierarchical models
  - Individual differences – next time
- We used a classic conjugate prior
  - Not always so easy; see C7 of F&L

3. Bayesian approaches

Why doesn't everyone use Bayesian methods?

1. Computational complexity
    - Conjugate priors aren't always appropriate
    - Inference can be computationally expensive

3. Bayesian approaches

Why doesn't everyone use Bayesian methods?

2. Convention, momentum, philosophical differences
    - More psychologists use/understand* frequentist methods
    - Out-of-the-box hypothesis tests are less work
    - Suspicion about priors

3. Bayesian approaches

Why doesn't everyone use Bayesian methods?

3. Technical barriers
    - Bayesian methods expose more mathematical detail
    - Until recently, few good tools for running non-trivial Bayesian analyses

But:

- Faster computers
- Friendlier/better tools, e.g. Stan
- Wider adoption and better dissemination
    - Bayesian analyses much more common than 5-10 years ago
    - Materials for wider audiences, e.g., Kruschke's "puppy book"

Summary

- MLE and MAP generate point estimates of the parameters
- Sensible priors can mitigate overfitting until MAP estimation
- Better yet: Bayesian methods – priors and integrating over parameters
  - less prone to overfitting and allows better use of informative priors
  - allows more questions to be answered more directly
- Conjugate prior distributions, where the prior and the posterior have the same form given a particular likelihood function
  - Easily-interpretable posteriors
  - Closed-form expressions for many quantities of interest