THE UNIVERSITY *of* EDINBURGH
informatics

Applied Machine Learning (AML)

Naive Bayes

Oisin Mac Aodha • Siddharth N.

Naive Bayes Classification

Generative Classification

- In classification the goal is to a learn a function $\hat{y} = f(\boldsymbol{x}; \boldsymbol{\theta})$
  - $y \in \{1, ..., C\}$ is one of C classes (e.g. spam / ham, digits 0-9)
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- We can use **Bayes's rule** to convert the class prior and class-conditionals to a posterior probability for a class

$$p(y = c|\boldsymbol{x}) = \frac{p(\boldsymbol{x}|y = c)p(y = c)}{\sum_{c'} p(\boldsymbol{x}|y = c')p(y = c')} \qquad \text{posterior} = \frac{\text{likelihood} \times \text{prior}}{\text{evidence}}$$

Generative Classification - Components

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  - The prior for each class
  - Encodes which classes are common and which are rare

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- $p(y = c)$: **Prior probability**
  - The prior for each class
  - Encodes which classes are common and which are rare
- $p(\boldsymbol{x})$: **Evidence**
  - $p(\boldsymbol{x}) = \sum_{c'} p(\boldsymbol{x}|y = c')p(y = c')$
  - Normalises the probabilities across observations
  - Does not impact which class is the most likely

Representing the Class Conditional Density

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- For example, for multivariate continuous data i.e. $\boldsymbol{x} \in \mathbb{R}^D$, we can use the multivariate Gaussian with parameters $\boldsymbol{\mu}_c$ and Σ_c
- However, this requires estimating $D(D+1)/2$ parameters for each class covariance matrix Σ_c, which may be problematic as the dimensionality D gets large

Naive Bayes Assumption

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- The model is called "*naive*" since we do not expect the features to be independent, even conditional on the class labels
- Even though this assumption is not typically true, Naive Bayes can still work well in practice

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i.e. once we know B, knowing C does not provide additional information about A

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- Hot weather "explains" all the dependence between the beach and heatstroke
- In classification, the class label explains all the dependence between the features

Conditional Independence of Features - Example

- Suppose we had a feature vector $\boldsymbol{x} = [x_1, x_2, x_3]^\top$, we can write out the conditional probability as

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- In Naive Bayes we make the following simplifying assumption

$$p(\boldsymbol{x}|y) = \prod_{d=1}^{D} p(x_d|y)$$

Naive Bayes with Binary Data

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id	email	status
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3	“review your account”	ham
4	“review us”	spam
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- We can fit a **Naive Bayes classifier** to this data so that we can classify new emails

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- For example, for a vocabulary with the following words:
  { ‘password’, ‘review’, ‘send’, ‘us’, ‘your’, ‘account’ }

  The email containing the text **“send us your password”** would be encoded as
  $\boldsymbol{x} = [1, 0, 1, 1, 1, 0]$

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  The email containing the text **“send us your password”** would be encoded as
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- We can exclude common words, e.g. ‘a’, ‘the’, ...

Representing Text Data

- Given the following vocabulary we can extract features from our data:
  { 'password', 'review', 'send', 'us', 'your', 'account' }

id	email	feature	status
1	"send us your password"	[1, 0, 1, 1, 1, 0]	spam
2	"send us review"	[0, 1, 1, 1, 0, 0]	ham
3	"review your account"	[0, 1, 0, 0, 1, 1]	ham
4	"review us"	[0, 1, 0, 1, 0, 0]	spam
5	"send your password"	[1, 0, 1, 0, 1, 0]	spam
6	"send us your account"	[0, 0, 1, 1, 1, 1]	spam

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- Here, $\phi_{dc} \in [0, 1]$ is the probability that $x_d = 1$ when y is class c

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- In the case of binary features, the maximum likelihood estimate is

$$\hat{\phi}_{\text{MLE}} = \frac{N_{dc}}{N_c}$$

i.e. the empirical fraction of times that feature d is present in examples from class c

Spam Classification Example

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- Class priors:

  $p(\text{spam}) = 4/6 \quad p(\text{ham}) = 2/6$

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$p(x_d \mid \text{spam})$	$p(x_d \mid \text{ham})$	x_d
2/4	0/2	password

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3/4	1/2	us
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Classifying New Data - Spam Likelihood

- Given an *new* email we would like to be able to classify it

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1/4	1/2	account

$$p(\boldsymbol{x}_t \mid \text{spam}) = p(0, 1, 0, 1, 0, 0 \mid \text{spam})$$
$$= (1 - \frac{2}{4})(\frac{1}{4})(1 - \frac{3}{4})(\frac{3}{4})(1 - \frac{3}{4})(1 - \frac{1}{4}) = 0.004$$

Classifying New Data - Ham Likelihood

- Given an *new* email we would like to be able to classify it
- For example, given the test email: "review us now"
- $\boldsymbol{x}_t = [0, 1, 0, 1, 0, 0]^\intercal$

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$$p(\boldsymbol{x}_t \mid \text{ham}) = p(0, 1, 0, 1, 0, 0 \mid \text{ham})$$
$$= (1 - \frac{0}{2})(\frac{2}{2})(1 - \frac{1}{2})(\frac{1}{2})(1 - \frac{1}{2})(1 - \frac{1}{2}) = 0.0625$$

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- Thus, according to our model, the probability that "review us now" is a ham email is $p(\text{ham}|\boldsymbol{x}_t) = 0.88$

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$$= \frac{0.0625 \times 2/6}{0.004 \times 4/6 + 0.0625 \times 2/6}$$
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- Thus, according to our model, the probability that "review us now" is a ham email is
  $p(\text{ham}|\boldsymbol{x}_t) = 0.88$
  and by extension,
  $p(\text{spam}|\boldsymbol{x}_t) = 1 - 0.88$

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  - Laplace smoothing: add a small positive number to all counts

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- **Independence assumption**

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  - Solution: never allow *zero* probabilities
  - Laplace smoothing: add a small positive number to all counts

$$p(x_d|y) = \frac{N_{dc} + \epsilon}{N_c + 2\epsilon}$$

- **Independence assumption**
  - Every feature contributes independently
  - e.g. you can fool Naive Bayes by adding lots of 'hammy' words

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Naive Bayes with Continuous Data

Continuous Feature Example - Task

- Task: Distinguish alpacas from llamas based on size

Continuous Feature Example - Data

- Task: Distinguish alpacas from llamas
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  - Features: height (cm) and weight (kg)
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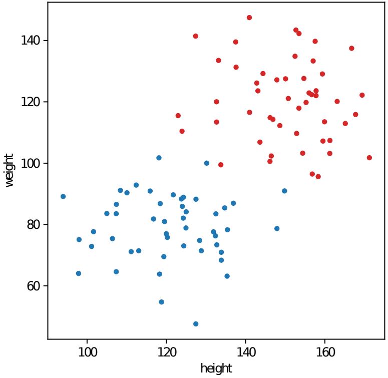
  - Assume height and weight are independent



Naive Bayes with Continuous Data

- In the case of real-valued features, $x_d \in \mathbb{R}$, we can use the univariate Gaussian distribution

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- This is equivalent to Gaussian discriminant analysis using *diagonal covariance matrices*

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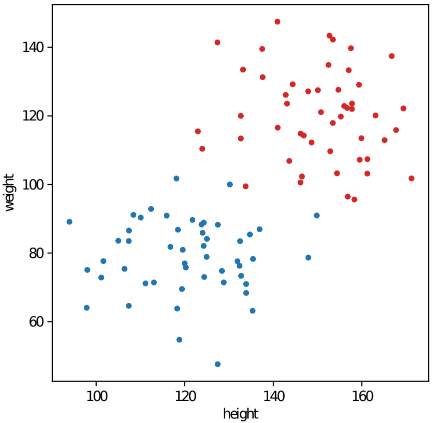

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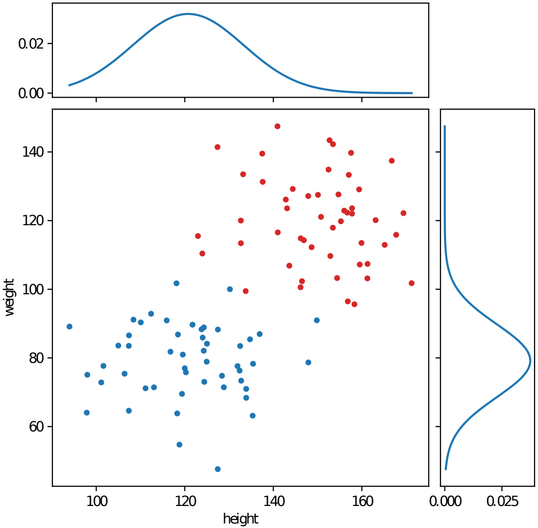

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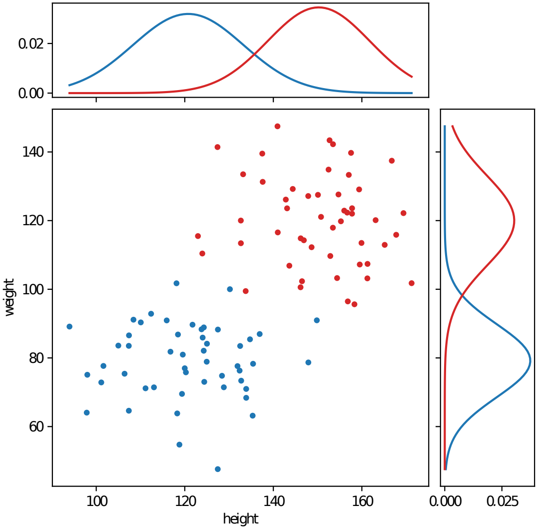
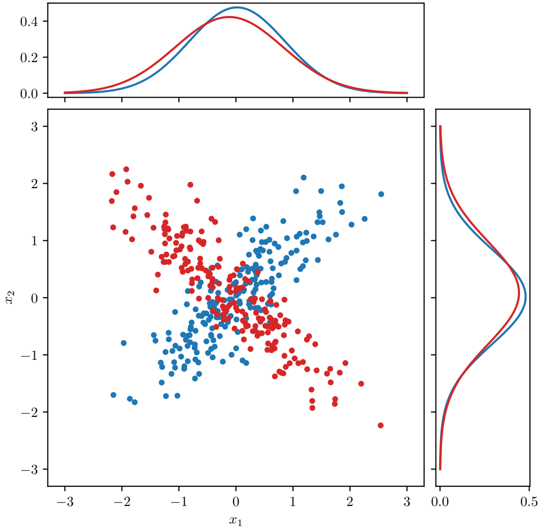


Problems With Naive Bayes

- The conditional independence assumption used by Naive Bayes can fail to capture relationships that may be present in some datasets



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- We presented the **Naive Bayes** classifier
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Summary

- We presented the **Naive Bayes** classifier
- It assumes that *features* are conditionally independent given the *class*
- This results in a reduction in the number of parameters we need to learn
- We can apply it to both *discrete* and *continuous* data
- This underlying assumption of Naive Bayes is a simplification that will not necessarily work for all datasets