WEEK 5

<u>Writing out proofs in tree format</u>. First consider a simple proof and the sequents represented by each line of it.

1	(1)	(P & Q) → R	A	(P & Q) → R ⊢ (P & Q) → R
2	(2)	P	A	P ⊢ P
3	(3)	Q	A	Q ⊢ Q
2,3	(4)	P & Q	2,3 &I	P, Q ⊢ P & Q
1,2,3	(5)	R	1,4 MPP	(P & Q) → R, P, Q ⊢ R
1,2	(6)	Q → R	3,5 CP	(P & Q) → R, P ⊢ Q → R
1	(7)	P → (Q → R)	2,6 CP	(P & Q) → R ⊢ P → (Q → R)

Now consider the sequents written out in the form of a tree according to the applications of the rules. The structure of the tree is recorded in the proof on the left above by the line numbers cited in the annotations.

```
                                             ────────── (A)          ────────── (A)
                                             P  ⊢  P                 Q  ⊢  Q
─────────────────────────── (A)              ──────────────────────────────── (&I)
(P & Q) → R  ⊢  (P & Q) → R                        P, Q  ⊢  P & Q
──────────────────────────────────────────────────────────────────── (MPP)
                  (P & Q) → R, P, Q  ⊢  R
                  ─────────────────────── (CP)
                  (P & Q) → R, P  ⊢  Q → R
                 ───────────────────────── (CP)
                  (P & Q) → R  ⊢  P → (Q → R)
```

The proof is in some ways clearer when written in this form. Here is another:

```
                            ──────────────(A)   ─────────(A)
                            P → Q  ⊢  P → Q      P  ⊢  P
──────────────(A)         ────────────────────────────(MPP)      ─────────(A)
P v Q  ⊢  P v Q                 P → Q, P  ⊢  Q                   Q  ⊢  Q
──────────────────────────────────────────────────────────────────────(vE)
                         P v Q,  P → Q  ⊢  Q
```

In discovering proofs in the tree format, it is easiest to start at the bottom, with the sequent to be proved, and explore ways in which it might have resulted from earlier sequents. Applications of CP are particularly easy in this format, while applications of MPP are relatively hard to spot! Notice that in proofs written out as trees the <u>applications</u> of the rules look like their abstract <u>specifications</u>. Pick out A, B, C, X, Y and Z of the definition of vE in the above example to see how the rule is working.

Deriving Rules

The tree format also makes it easy to show certain rules derivable. This is done by exhibiting a way of getting from the "inputs" of a suggested rule to its "output". The following examples illustrate.

1. $$\frac{X : A \qquad Y, A : B}{X, Y : B}$$

Derivation

$$\dfrac{X : A \qquad \dfrac{Y, A : B}{Y : A \to B}\,(CP)}{X, Y : B}\,(MPP)$$

2. $$\frac{X : A \qquad Y, B : C}{X, Y, A \to B : C}$$

Derivation

$$\dfrac{\dfrac{Y, B : C}{Y : B \to C}\,(CP) \qquad \dfrac{X : A \qquad \dfrac{}{A \to B : A \to B}\,(A)}{X, A \to B : B}\,(MPP)}{X, Y, A \to B : C}\,(MPP)$$

3. $$\frac{X, A, B : C}{X, A \,\&\, B : C}$$

Derivation

$$\dfrac{\dfrac{\dfrac{}{A \,\&\, B : A \,\&\, B}\,(A)}{A \,\&\, B : B}\,(\&E) \qquad \dfrac{\dfrac{\dfrac{}{A \,\&\, B : A \,\&\, B}\,(A)}{A \,\&\, B : A}\,(\&E) \qquad \dfrac{\dfrac{X, A, B : C}{X, A : B \to C}\,(CP)}{X : A \to (B \to C)}\,(CP)}{X, A \,\&\, B : B \to C}\,(MPP)}{X, A \,\&\, B : C}\,(MPP)$$

LOGIC 1 EXERCISE SHEET 4 1986-7

Monday and Wednesday tutorial groups: HAND IN solutions at your next tutorial (in week 6). Thursday and Friday groups: you need NOT hand in solutions.

Prove these sequents. You may use all of the primitive rules, including RAA, but not TI or SI. Note that the last three of these sequents are rated as "difficult".

1. $P \to Q \vdash (Q \to -P) \to -P$

2. $P \to R,\ Q \to -R \vdash -(P \& Q)$

3. $\vdash P \lor -P$

4. $\vdash P \lor (P \to Q)$

✓ 5. $P \to (Q \lor R),\ R \to S \vdash P \to (Q \lor S)$

✓ 6. $P \lor Q,\ P \lor R \vdash P \lor (Q \& R)$ See Tutorial extra sheets

7. $-(P \to Q) \vdash P \& -Q$

8. $\vdash (P \to Q) \lor (Q \to R)$

These are LEMMON-AID's sheet E4. Sheet S4 is more of the same and is useful for practice. Sheet X4 is in the same general vein, but contains some "very difficult" exercises - especially its last three. If you can successfully complete sheet X4 then you have nothing more to learn about doing proofs in the propositional part of Lemmon's calculus.

LOGIC 1 TUTORIAL EXERCISE 1986-7

WEEK 7

Suppose we add to the language of Lemmon's system of logic a new dyadic truth-functional connective * . Thu rules for the new connective in proofs are:

*I.

$$\frac{X, A \; : \; C \qquad Y, B \; : \; -C}{X, Y \; : \; A * B}$$

Cite 4 numbers: the 2 input line numbers and the 2 discharged assumption numbers.

*E.

$$\frac{X \; : \; A * B \qquad Y \; : \; A \qquad Z \; : \; B}{X, Y, Z \; : \; C}$$

Cite the 3 input line numbers.

1. What is meant by calling *
   (i) a connective?
   (ii) dyadic?
   (iii) truth-functional?

2. Prove the following sequents, using Lemmon's rules together with *I and *E as given above (but not TI or SI).

   (a) $P \rightarrow (Q \rightarrow (R \,\&\, -R)) \vdash P \rightarrow -Q$

   (b) $P * Q \vdash P \rightarrow (Q \rightarrow (R \,\&\, -R))$

   (c) $-P \vee -Q \vdash P * Q$ [hint: $-P \vee -Q, Q \vdash -P$]

3. (a) Give the truth table for * as defined in Question 1. Briefly explain how you arrived at your answer.

   (b) Find formulas <u>in which * is the only connective</u> with the same truth tables as -P, P & Q and P v Q.