PRELIMINARIES

This is a "fat" half course, running for 12 weeks (until week 3 of the Spring Term). In the Arts Faculty it is the first half of Logic and Philosophy of Science 1. In other faculties it is a self-contained half course. There are no prerequisites for Logic 1h.

Classes. Three lectures per week: Monday, Tuesday, Thursday at 5pm in Lecture Theatre 3, Appleton Tower. Tutorials weekly in groups of about 10 at times and venues to be arranged. Some additional (optional) classes are likely. Revision classes will be held later in the year.

Assessment. Class examination (2 hours) on January 29th. Degree examination (also 2 hours) in June. Merit certificates are awarded on the basis of the class exam. with other class marks taken into account in borderline cases. There are no exemptions from the degree examination.

Due Performance. The work to be duly performed for this course consists of:

(a) Attendance at tutorials. THIS IS COMPULSORY. THIS MEANS YOU.

(b) Handing in answers to exercises, as required from time to time, not unreasonably or persistently later than the due date.

(c) Sitting the class examination.

Failure to complete this modest workload will result in witholding of certificates of due performance unless a good reason is produced. Pressure of work from other courses is not a good reason. If you miss classes through illness, get a medical certificate. Fair warning will be given in writing to any student in danger of losing a "d.p.".

Set Book. We shall not be working through a text in this course. The system and notation presented are taken (with very few modifications) from:

E. J. Lemmon,
Beginning Logic,
Van Nostrand Reinhold, London, 1965 (or later editions).

Students in the past have found access to a copy of Lemmon useful though not absolutely essential. It is available in paperback from Thin's.

Computer Program. There is a program called LEMMON-AID which checks proofs for validity and is intended to go with Logic 1h. It runs on the University's SIRIUS microcomputers and can be used by any Logic 1 student during any open session of the Micro Lab in the Appleton Tower. Guidance on the use of LEMMON-AID will be given at an appropriate point in the course.

Help. Any Logic 1 student wanting help or advice on any aspect of the course (or other matters within reason!) can call at my room in the David Hume Tower at any time between 8.30 am and 6.30 pm, Monday to Friday. I am always available for consultation except when actually taking classes.

October 1986

Dr J K Slaney

Room 2.05, DHT.

SHEET 1

The subject of study in this course is formal logic. It is easier to illustrate what logic is than to give a simple definition, but philosophy courses traditionally start by defining the topic, so at this stage a definition may be in order. Logic, then, is the science of reasoning. It has to do not with the psychology of reasoning - it is not science in that sense - but with the possible inferences which can be drawn, with the language in which they can be expressed and with their correctness or incorrectness. Expansion and explanation of this statement will have to await the development of the study.

Logic has to do with arguments. An "argument", as logicians use the term, is not a dialogue but a series of statements called the premisses of the argument, followed by some word like 'so' or 'hence' or 'therefore', followed by another statement called its conclusion. The premisses are supposed to lend rational support to the conclusion in the sense that anyone who accepts the premisses and is faced with the argument is forced, on pain of irrationality, to accept the conclusion. As an example of an argument, consider:

No dogs are allowed in here.	(premiss)
Snoopy is a dog.	(premiss)
So Snoopy is not allowed in here.	(conclusion)

As a more complex case, consider:

1. The deed was done by either his Lordship, the butler or the Unfortunate Miss Lavinia.
2. Whoever did the deed was in the house on Friday, but the Unfortunate Miss Lavinia did not arrive until Saturday.
3. Therefore (from 1 and 2) the deed was done by either his Lordship or the butler.
4. Whoever did the deed must have been sober at the time, which clearly rules out his Lordship.
5. Therefore (from 3 and 4) the butler dunnit.

Here the premisses of the argument are statements 1, 2 and 4. The conclusion is statement 5. We get from the premisses to the conclusion by way of two smaller arguments, the conclusion of the first (statement 3) forming one of the premisses of the second. Such an intermediate stage in a compound argument may, if it is important enough, be dignified with the title "lemma". It is not strictly necessary, for 5 follows from 1, 2 and 4 without it; it is added only to make the argument easier to follow. Later in the course we shall encounter many argument structures much more complex than this one, and it is part of our aim to develop the means to handle them with ease and efficiency.

A good argument is said to be valid. An argument is valid if, and only if it is impossible that its premisses should all be true while its conclusion is false. That is, if the premisses are true, then (necessarily) so is the conclusion. To assert the premisses of a valid argument and deny its conclusion is to contradict oneself. Conversely, an argument is invalid if there is a way for its premisses to be true and its conclusion false.

Notice the difference between validity and truth. Arguments are valid or invalid; statements or sentences or propositions are true or false.

The premisses of a valid argument are said to entail or imply its conclusion. The conclusion is said to follow from, be deducible from, be derivable from or be a consequence of the premisses.

A valid argument need not, of course, have true premisses. We require only that *if* the premisses are true then so is the conclusion. An argument will be called *sound* just in case (a) it is valid, and (b) all its premisses are true. Note that Lemmon in the textbook uses the word 'sound' to mean 'valid'. The usage just specified is standard, however, so we shall just let Lemmon differ.

Consider now the following valid argument.

> Snoopy is a cat.
> All cats are reptiles.
> Therefore Snoopy is a reptile.

Clearly not only is this valid (though unsound), but we can explain *why* it is valid. It is of a valid *form*, which we can express using letters to stand in for predicates and names somewhat as they stand for numbers in algebra:

> a is F.
> Everything F is G.
> Therefore a is G.

Notice that reasonably idiomatic English sentences may have to be paraphrased to make them fit the form. For this process we have to rely on our educated intuitions as speakers of English: there is no automatic method available.

Logic gives us valid *argument forms*. A form is valid if and only if every argument of that form is valid. To show that an argument is valid we typically find a valid form which it exemplifies. A given argument will be an instance of more than one form, so it is very difficult to show *invalidity* by means of pure logic. We can show an argument *form* to be invalid by finding an argument of that form with actually true premisses and an actually false conclusion. This is called a *counter-example* to the invalid form. Even if we can't find a valid form for an argument, it may still *be* valid.

> The People's Flag is red.
> Therefore the People's Flag is coloured.

is a valid argument, even though it is likely to defeat our logical system. Formal logic will not, then, capture *all* valid arguments. But it captures a large class of them, and is our tool for investigating the concept of validity.

We are now going to concentrate on the branch of logic known as the *sentential* or *propositional* calculus. This involves only whole sentences and the logical relationships between forms of combination of whole sentences. We shall cease (for now) to worry about particles like 'all' and 'some' and about such things as names and predicates. Instead we shall consider such locutions as

> It is false that......
> Either......or......
> If......then......

These are called *connectives*. A connective is an expression which applies to one or more sentences to form a longer sentence in which the originals function as parts. Natural languages like English abound with connectives, like:

> The Ancient Greeks believed that......
> ......because......
> ......although......
> Probably......
> Maybe......
> I find it incredible/disgusting/exciting/etc. that......

Try filling the blanks with various typical English sentences like

The square root of 2 is irrational.
Hearts never actually win so much as a plastic teaspoon.
Some people get pleasure from Logic 1h.
Pigs will fly.

to gain some feel for the way connectives work in English. We shall study just a few connectives whose logical properties are particularly interesting. These, with the notation we shall use for them, are:

Both ...... and	.... & ....
Either ...... or	.... v ....
If ...... then	.... → ....
It is not the case that	- ....
...... if and only if	.... ↔ ....

With this notation, starting from sentence letters, we can build up sentence forms of any complexity:

-P → Q
(P&Q) v (P&R)
P → (Q → (R → S))   etc.

Notice that we use parentheses in the familiar way to disambiguate compounds. Just as in arithmetic (3 x 4) - 1 differs from 3 x (4 - 1), so in logic we must distinguish between (P & Q) → R and P & (Q → R). The symbol '-' is always read as applying to the smallest following sentence, so that for instance -P v Q is read as (-P) v Q rather than -(P v Q). If we want to express the latter we have to parenthesise. Sentence forms built up in this way with the given formal connectives shall be called <u>formulas</u>. A precise definition of "formula" will be given later.

Before developing the formal calculus of logic, we should note two important concepts concerning occurrences of connectives in formulas. First, every occurrence of a connective has a <u>scope</u>. This consists of the connective itself together with the formulas it connects. Lemmon's definition of the scope is <u>the shortest (sub)-formula in which the occurrence lies</u>. For example, consider the formula

-(P & Q) → ((P v R) → -S)

The scope of the first '-' is -(P&Q), while the scope of the second '-' is -S. The scope of the '&' is P&Q, and that of the second '→' is (PvR) → -S.

Second, the <u>main connective</u> of any formula is <u>the occurrence of a connective which is not inside the scope of any other.</u> That is, the scope of the main connective is the entire formula. Thus the main connective of the above sample formula is the first '→'. The main connective of

P [&] (Q & R)

is the first '&', while the main connective of

(P & Q) [&] R

is the second '&'. All the rules of our formal calculus will operate on main connectives only, so this concept, though simple, is very important.

<u>It is convenient to have a short notation to represent the claim that a given argument form is demonstrably valid.</u> For this we use the symbol " $\vdash$ ", (the "turnstile", sometimes misleadingly called the "assertion sign"). Suppose that A and B are any two formulas. Then

A therefore B

is an argument form. If we want to claim that we can prove it valid, we write

$A \vdash B$.

More generally, where $A_1, \ldots\ldots, A_n$ and B are all formulas,

$A_1, \ldots\ldots, A_n : B$ is a <u>sequent</u>, and

$A_1, \ldots\ldots, A_n \vdash B$ means that this sequent can be <u>proved</u>.

By a slight abuse of language, I shall sometimes call $A_1, \ldots\ldots, A_n \vdash B$ the "sequent", in accordance with Lemmon's usage (which is again slightly non-standard on this point). By another fairly harmless deviation, a correct sequent will be said to be "valid" rather than "true". Thus for example,

P & Q : P
P, Q : P & Q
P : P

are all valid sequents (because they represent valid argument forms), while

P : P & Q

is invalid (why?).

One more bit of terminology can conveniently be introduced at this point. One formula is said to be a <u>substitution instance</u> of another if and only if every sentence of the first form is also of the second. An alternative definition is this: formula A is a substitution instance of formula B if and only if A results from B by substitution of formulas for sentence letters. Analogously, and importantly, sequents can be substitution instances of other sequents. As a special case, notice that every formula is a substitution instance of itself. To illustrate, (P&Q) v -R is a substitution instance of Q v P, resulting from it by substitution of P&Q for Q and -R for P. The same formula is <u>not</u> a substitution instance of Q v Q, because substitution must be <u>uniform</u>, the same formula replacing the same letter throughout. This definition of "substitution instance" is specific to propositional logic. When we later come to more intricate parts of logic it will no longer be adequate.

Check that you know the meanings of the following:

argument,	valid,	sound,
argument form,	counter-example,	connective,
sentence letter,	formula	scope,
main connective,	sequent,	substitution instance

SHEET 2

We shall now start to construct a formal calculus for the rigorous proof of logically valid sequents. First we must say what is to count as a proof in the system, and to explain this we begin with the notion of a derivation. A derivation is produced by filling out an argument with intermediate steps between premisses and conclusion, perhaps with comments saying how each line follows from earlier ones. By means of derivations, dificult and unobvious arguments (for example, those relating the axioms of geometry to its theorems) can be spelled out and reduced to small, simple, obviously valid steps. The proof of sequents in formal logic is likewise reduced to about a dozen small-scale rules whose operation can be iterated to produce intricate derivations.

A proof consists of lines of proof, written one below the other and numbered consecutively starting from 1. No great significance attaches to the choice of numerals as labels: anything similar, such as letters a,b,c,.... would have served equally well. The numerals just happen to be a convenient and familiar sequence. To the right of the line number is written a formula, and to the left is written zero or more numerals separated by commas. This represents a sequent. the written formula being its conclusion and the "assumption numbers" being shorthand for the premisses. For example,

1,2,5 (16) $P \rightarrow (Q \& R)$

would be the 16th line of a proof and means that $P \rightarrow (Q\&R)$ has been derived from the premisses assumed on lines 1, 2 and 5. The line is also annotated by the addition to the right of a brief justification for the inclusion of that line. Details will be specified when we come to particular cases. In order to count as a proof, a series must have every constituent line justified.

The most elementary rule is what Lemmon calls the Rule of Assumptions.

> At any stage of any proof, any formula may be introduced on a new line. It depends on that line number only (i.e. on itself) as assumption. The annotation is the letter "A".

Evidently, the rule of assumptions only allows us to prove sequents which are substitution instances of P : P. All such sequents are clearly valid, but rather unexciting. To make logic more interesting, we need rules governing the behaviour of various connectives. These will transform the trivially valid results of the rule of assumptions into nontrivially valid sequents, some of which may even be surprising.

The simplest connective to grasp is conjunction, &. This is intended to corespond to the English word 'and', though it will be clear that this correspondence is not exact. For one thing, 'and', like its cognates in other Indo-European languages, does not only join sentences. Sometimes it joins nouns, verbs, adjectives, etc. as in:

Jack and Jill went up the hill.	(joining proper nouns)
Dogs delight to bark and bite.	(joining verbs)
I'm ready and willing if you are.	(joining adjectives)

It may also join adverbs or phrases of almost any kind. In the second place, unpacking these constructions into conjunctions of the form P&Q may give incorrect results. While the second example above does mean

Dogs delight to bark and dogs delight to bite

we cannot similarly convert

Chalk and cheese are different.
York is between London and Edinburgh.
Tom, Dick and Harry carried a piano upstairs.

into

Chalk is different and cheese is different.
York is between London and York is between Edinburgh.
Tom carried a piano upstairs, so did Dick and so did Harry.

In the third place, even where 'and' really is a connective, it may not be &.

His wife left him and he cooked the supper.
He cooked the supper and his wife left him.

are not synonymous, for instance, while

One false move and I shoot!

is not a conjunction at all, but a conditional.

For all that, 'and' is relatively well-behaved logically. The other connectives are in their various ways even harder to formalise adequately. The formal rules for & are very simple. A conjunction P&Q carries exactly the force of its two conjuncts P, Q taken together. Thus we may validly infer the conjunction from the conjuncts and, conversely, the conjuncts from it:

$$\frac{P \,\&\, Q}{\therefore\ P} \qquad \frac{P \,\&\, Q}{\therefore\ Q} \qquad \frac{\begin{matrix} P \\ Q \end{matrix}}{\therefore\ P \,\&\, Q}$$

The general rules for & follow immediately. Where A and B are any formulas and X and Y any sets of formulas (represented by lists of assumption numbers)

$$\frac{X \ :\ A \,\&\, B}{X \ :\ A} \qquad \frac{X \ :\ A \,\&\, B}{X \ :\ B}$$

$$\frac{X \ :\ A \qquad\qquad Y \ :\ B}{X,Y \ :\ A \,\&\, B}$$

The first two of these are annotated with the one line number (of the input line) and '&E' ("& Elimination"). The last is annotated with the two input line numbers and '&I' ("& Introduction"). The order of the two input lines in the proof is irrelevant. For brevity and readability, we can write rules like these which do not <u>change</u> the stock of assumption numbers with the "X"s and "Y"s omitted:

$$\frac{A \,\&\, B}{A} \qquad \frac{A \,\&\, B}{B} \qquad \frac{A \qquad B}{A \,\&\, B}$$

Examine the following proofs to see how the two rules operate. Try to see how each application of &E or &I is an instance of the abstractly stated rules.

$P \& Q \vdash Q \& P$

1	(1)	P & Q	A
1	(2)	P	1 &E
1	(3)	Q	1 &E
1	(4)	Q & P	2,3 &I.

$P \& Q,\ R \& S \vdash P \& S$

1	(1)	P & Q	A
2	(2)	R & S	A
1	(3)	P	1 &E
2	(4)	S	2 &E
1,2	(5)	P & S	3,4 &I.

Exercise: write out the sequent represented by each line of this proof.

The next connective to be introduced is the <u>conditional</u>, →. As noted earlier, this is intended to be read 'if...then...', so we begin by considering the meaning if that expression in English. Under what circumstances is it correct to assert 'If P then Q'? I suggest that such an assertion is correct if (and only if), given the facts, Q follows from P. That is to say, $P \rightarrow Q$ is true iff there is a valid argument whose conclusion is Q and whose premisses are P and (perhaps) some truths. This definition of the truth conditions for $P \rightarrow Q$ has two halves. First, if you are in a position to assert $P \rightarrow Q$, then you are in a position to argue from P to Q. This is obvious, since you are in a position to use the argument form

$P \rightarrow Q,\ P$ therefore Q.

Second, if there is available enough collateral information to get you from P to Q, then you may correctly assert the conditional $P \rightarrow Q$.

Given the suggested account of conditionals, we should expect a formal logic which deals with a connective → intended as 'if...then' to satisfy the condition that for all formulas A and B, and for any set X of assumptions,

$$X \vdash A \rightarrow B \quad \text{iff} \quad X, A \vdash B.$$

This is the <u>Deduction Equivalence</u> and lies at the heart of logic as we are doing it. Its importance is that it ties the <u>connective</u> → to the <u>relation</u> of deducibility. It quickly gives rise to the two rules governing the behaviour of → in Lemmon's system. The first is "Modus Ponendo Ponens":

$$\frac{A \rightarrow B \qquad A}{B}$$

Annotation is the two input line numbers, and the expression 'MPP'.

or in unabbreviated format,

$$\frac{X : A \rightarrow B \qquad Y : A}{X, Y : B.}$$

The second is the rule of "Conditional Proof":

X,A : B / X : A → B	Annotation is the input line number, the discharged assumption number, and the expression 'CP'.

Note that CP has only <u>one</u> input line, the line where the consequent of the desired conditional occurs, and that the antecedent A must be one of the <u>assumptions</u> on which it depends. That assumption number disappears as a result of CP. It is said to be "discharged".

The following proofs illustrate the use of the rules.

P → R ⊢ (P & Q) → R

1	(1)	P → R	A	(premiss)
2	(2)	P & Q	A	(the antecedent assumed)
2	(3)	P	2 &E	
1,2	(4)	R	1,3 MPP	(the consequent derived)
1	(5)	(P & Q) → R	2,4 CP.	

Note that line (5) does <u>not</u> come from lines (2) and (4). It comes from line (4) alone, discharging assumption 2. In the application of CP here, X is the numeral '1'. A is the formula P & Q, represented on the left of line (4) by the numeral '2'. B is the formula R. With those identifications, check that we do indeed have an instance of the rule as formally stated above.

An application of CP does not usually just "happen", like &E or MPP. It has to be set up by prior assumption of the intended antecedent. Where several CP steps are needed, the process is iterated, as in this proof:

P ⊢ (Q → R) → (Q → (P & R))

1	(1)	P	A	(premiss)
2	(2)	Q → R	A	(antecedent for CP)
3	(3)	Q	A	(antecedent for CP)
2,3	(4)	R	2,3 MPP	
1,2,3	(5)	P & R	1,4 &I	(first consequent)
1,2	(6)	Q → (P & R)	3,5 CP	(discharging 3)
1	(7)	(Q → R) → (Q → (P & R))	2,6 CP.	

Here the desired conclusion is a conditional, so we assume its antecedent at line (2). We are then trying to get its consequent, Q → (P & R) so that CP can apply. That is, the sub-goal is to prove the sequent

P, Q → R ⊢ Q → (P & R)

which we achieve at line (6). The new conclusion (the consequent of the old one) is in turn a conditional, so it too will be proved by Conditional Proof. So we assume <u>its</u> antecedent, Q, at line (3) and aim for its consequent P & R from the three assumptions now in play. That is, the new sub-sub-goal is

P, Q → R, Q ⊢ P & R

which is derived at line (5) by a couple of easy moves.

The theory of the conditional is so important, and the operation of Conditional Proof so critical for the formal system being presented in this course, that you must make sure you understand it. The computer program LEMMON-AID has dozens of CP exercises designed to help on just this point.

SHEET 3

An application of CP always discharges one of the assumptions. Now what happens if we carry on applying the rule when there is only one assumption left? Well, the left-hand side of the sequent produced is, naturally enough, empty:

1	(1)	P & Q	A
1	(2)	P	1 &E
	(3)	(P & Q) → P	1,2 CP.

A formula thus proved from the empty set of premisses is called a <u>theorem</u> of logic. As a special case of the deduction equivalence, A → B is a theorem if and only if A ⊢ B. (I.e. iff the sequent A : B is provable). We write

⊢ A

to claim that the formula A is a theorem. Theorems may be used in the proof of sequents with non-empty premiss sets, as in this example:

1	(1)	(P → P) → Q	A
2	(2)	P	A
	(3)	P → P	2,2 CP
1	(4)	Q	1,3 MPP.

Recall the informal definition of entailment (validity of arguments) with which we began. An argument is valid iff there is no way all its premisses could be true without its conclusion being true. It follows easily from this definition that whatever follows logically from <u>some</u> of the premisses also follows from the whole set. This is because, obviously, there is no way for <u>all</u> of the premisses to be true without in particular the ones needed for the argument being true. Hence if we can derive A from premisses X, then we can derive A from premisses X and Y together:

If X ⊢ A then X,Y ⊢ A.

This intuitively correct principle goes under several names in the standard literature of logic. Here it shall be called <u>augmentation</u>. If our system of logic is correct, it had better admit the principle in the sense that any augmentation of a provable sequent is also provable. This is indeed the case for finite sequents (and Lemmon's system has no place for infinite ones). For a typical example, consider the following proof:

P, Q, R ⊢ Q

1	(1)	P	A	(premiss)
2	(2)	Q	A	(premiss)
3	(3)	R	A	(premiss)
1,3	(4)	P & R	1,3 &I	
1,2,3	(5)	(P & R) & Q	2,4 &I	
1,2,3	(6)	Q	5 &E.	

The general method for augmentation is to conjoin the "dummy" premisses with &I, to use &I again to conjoin them onto a "working" premiss, and then to use &E to take them off again, leaving their assumption numbers on the left and all else unchanged. As a special instance of augmentation we have

$$\frac{\vdash A}{X \vdash A}$$

That is, any theorem is a consequence of anything you like, however irrelevant. This may be intuitively surprising, but on our working definition of validity it is well justified: if A is true no matter what, then in particular it is true whenever all the formulas in X are true.

It is now time to give a more rigorous definition of what is to count as a formula. This is done in stages. Firstly, the <u>alphabet</u> to be used consists of 13 symbols:

5 <u>sentence letters</u>	'P'	'Q'	'R'	'S'	'T'
1 <u>monadic connective</u>	'-'				
4 <u>dyadic connectives</u>	'&'	'v'	'→'	'↔'	
3 other symbols	')'	'('	'′'		

Secondly, an <u>atom</u> (or "atomic formula") is defined as follows:

1. Any sentence letter is an atom.
2. Where A is an atom, so is A followed by '′'.
3. Nothing is an atom except as a consequence of clauses 1 and 2 above.

So an atom is just a sentence letter followed by zero or more "primes". Thirdly, there is a definition in similar style of "formula" in terms of "atom".

1. Any atom is a formula.
2. Where A is a formula, so is a monadic connective followed by A.
3. Where A and B are formulas, so is '(', followed by A, followed by a dyadic connective, followed by B, followed by ')'.
4. Nothing is a formula except by clauses 1, 2 and 3 of this definition.

In writing out formulas we allow ourselves a few freedoms such as omitting the outermost pair of parentheses. This makes for readability. Following the definition, we can show whether a given string of symbols is a formula or not. A formula is assembled from sentence letters in steps, each step producing a sub-formula either by extending an atom (by adding "primes") or by introducing a connective as the main connective of that sub-formula. Work through the definition, constructing a formula like

(--P → (-(Q v P) & R))

to make sure you understand how "formula" has just been defined. Nowhere else in Logic 1 (and probably nowhere else in life) do we pay quite so much attention to quite such minute details. Logic has this way of making us examine, and be precise about, utterly simple things.

SHEET 4

The <u>negation</u> of a proposition is what is asserted when that proposition is denied. In logic we use the symbol '-' to turn the statement that so-and-so is the case into the statement that so-and-so is <u>not</u> the case. We read -P as 'It is not the case that P' and think of it as having the opposite force from P. It is easy to set out a table showing the conditions under which the use of negation results in a true statement or a false one:

P	-P
true	false
false	true

That is, negation reverses truth value. Some of the salient features of negation can be read off this table. Re-negating the negation of P reverses the truth value again, so the value of --P is the same as that of P. The equivalence of --P to P is called the <u>law of double negation</u>. It gives rise to a pair of formal rules:

$$\frac{--A}{A} \qquad\qquad \frac{A}{--A}$$

The annotation for each of these is the one input line number and 'DN'. In neither direction does DN have any effect on the pool of assumption numbers, so the usual 'X : ....' has been omitted for clarity. It is also worth noting that like all the rules DN only allows introduction and elimination of connectives in <u>main</u> position (well, two of them at once, if we must be pedantic). Occurences of the string '--' in minor positions cannot be added or deleted by DN.

DN on its own is not sufficient to capture negation. To complete the theory we add another rule, MTT (Modus Tollendo Tollens). This rule corresponds to the valid argument form

If P then Q.
Not Q.
Therefore not P.

Arguments of this form are common. Consider:

If it's Tuesday, this is Belgium.
This is not Belgium.
Therefore it isn't Tuesday.

If I stirred my coffee then the spoon is wet.
The spoon is not wet.
Therefore I did not stir my coffee.

The formal statement of the rule is, in brief format,

$$\frac{A \rightarrow B \qquad -B}{-A}$$

or in expanded format:

X : A → B	Y : -B
X,Y : -A	

The annotation is the two input line numbers and the expression 'MTT'. See how the rules look in practice by studying the following proof.

-P → -Q ⊢ Q → P

1	(1)	-P → -Q	A
2	(2)	Q	A { Antecedent of Q → P for CP }
2	(3)	--Q	3 DN
1,2	(4)	--P	1,3 MTT
1,2	(5)	P	4 DN
1	(6)	Q → P	2,5 CP.

Notice that the DN moves are essential if the proof is to accord with the rules, for Q is <u>not</u> the negation of -Q. At this point, it will be worthwhile to read sections 1 and 2 of Chapter 1 of Lemmon's book. There are useful exercises on negation at the end of section 2, and several proofs with notes in the course of the text.

Consider the following proof.

P → -P ⊢ -P

1	(1)	P → -P	A
2	(2)	P	A
3	(3)	P → P	A
2,3	(4)	P	2,3 MPP
2	(5)	(P → P) → P	3,4 CP
1,2	(6)	-P	1,2 MPP
1,2	(7)	-(P → P)	5,6 MTT
1	(8)	P → -(P → P)	2,7 CP
	(9)	P → P	2,2 CP
	(10)	--(P → P)	9 DN
1	(11)	-P	8,10 MTT.

This has a certain formal impressiveness, like a five-move combination in chess, but it achieves the effect by being devious, by turning in and in on itself until the conclusion pops out like a rabbit from a hat. Such obscure combinations are a fair part of the game, of course, but if the object of the system is to make proof-discovery <u>easy</u> then there ought to be a way of circumventing them with less virtuosity. Lemmon provides such a way in the form of an extra rule, <u>Reductio ad Absurdum</u>, RAA.

RAA rests intuitively on two principles. First, self-contradictions are logically false. That is, any statement of the form P & -P just logically <u>cannot</u> be true. Secondly, whatever entails a false conclusion must itself be false. Putting these two together, if some premisses entail a contradiction then at least one of the premisses must be false. So if P together with other premisses entails a contradiction Q & -Q then if those other premisses are all true, it must be P that is false. The formal rule corresponding to this thought is

X,A : B & -B
X : -A

Annotation: discharged assumption number, line number of contradiction, and 'RAA'.

RAA yields a powerful proof strategy. To derive a negative conclusion -A, assume the positive A and try for a contradiction. Also, where there is no obvious direct route to a desired conclusion, we can often get it by assuming the <u>negation</u> of what we eventually desire, deriving a contradiction and then applying RAA followed by DN. The RAA strategy is known as <u>indirect proof</u> or <u>proof by contradiction</u>. The following are simple examples.

P → -P ⊢ -P

1	(1)	P → -P	A	
2	(2)	P	A	
1,2	(3)	-P	1,2 MPP	{ Just line (6) of the old proof }
1,2	(4)	P & -P	2,3 &I	
1	(5)	-P	2,4 RAA.	

⊢ -(P & -P)

1	(1)	P & -P	A
	(2)	-(P & -P)	1,1 RAA.

As was proved in Lecture 10, the rule RAA is in principle redundant: its effect can be secured (though at greater length) by (mostly) MTT and DN. There will be more on the subject of such "derivable" rules in the next set of notes.

The next connective to be introduced is disjunction, '...or...'. Where A and B are formulas, A v B is the <u>disjunction</u> of A and B, which are its <u>disjuncts</u>. It is read 'Either A or B or both'. This is "inclusive" disjunction, and contrasts with the "exclusive" disjunction 'Either A or B but not both', which we rarely need in logic. So a disjunction is true iff at least one of its disjuncts is true. Notice that we may often be justified in <u>asserting</u> a disjunction without being justified in asserting either disjunct. For example unless we knew that the roulette wheel was going to stop on "either red or black", betting on the outcome would be even less rational than it actually is; but if we thereby knew <u>which</u> it would stop on, roulette would be even more boring than it actually is, and casino owners would be much poorer than they actually are. The rule for introducing v follows from inclusiveness:

$$\frac{A}{A \vee B} \qquad\qquad \frac{B}{A \vee B}$$

Annotation is the one input line number and 'vI'.
The elimination rule is more complicated. It is best approached by way of a simpler-looking rule corresponding to a valid form of argument:

P v Q, P → R, Q → R therefore R.

Arguments of this form are plentiful. Consider, for example:

Either inflation will rise or output will fall.
If inflation rises, the recession will deepen.
If output falls, the recession will deepen.
Therefore the recession will deepen.

Number x is either odd or even.
If x is odd then x(x+1) is even.
If x is even then x(x+1) is even.
Therefore x(x+1) is even.

In formal notation, the corresponding rule is of course

$$\frac{A \vee B \qquad A \to C \qquad B \to C}{C}$$

or, putting in the sets of assumption numbers,

$$\frac{X : A \vee B \qquad Y : A \to C \qquad Z : B \to C}{X,Y,Z : C}$$

Lemmon's version of vE is slightly different in that he replaces the two conditional inputs by forms equivalent to them by the deduction equivalence:

$$\frac{X : A \vee B \qquad Y,A : C \qquad Z,B : C}{X,Y,Z : C}$$

The annotation for vE is <u>five</u> numbers and the expression 'vE'. The five are the three input line numbers and the two discharged assumption numbers. That is, you cite

the line number of the disjunction
the assumption number of A and the line where C was derived from it
the assumption number of B and the line where C was derived from <u>it</u>.

This looks complicated, and in fact the best way into it is by working examples.

Example 1. P v Q ⊢ P v (Q v R).

1	(1)	P v Q	A	
2	(2)	P	A	{ Left disjunct of (1) for vE }
2	(3)	P v (Q v R)	2 vI	{ conclusion derived from it }
4	(4)	Q	A	{ Right disjunct similarly }
4	(5)	Q v R	4 vI	
4	(6)	P v (Q v R)	5 vI	{ conclusion derived again }
1	(7)	P v (Q v R)	1, 2,3, 4,6 vE.	

Example 2. P → -Q, P → -R ⊢ (Q v R) → -P.

1	(1)	P → -Q	A	
2	(2)	P → -R	A	
3	(3)	Q v R	A	{ Antecedent for CP }
4	(4)	Q	A	{ Left disjunct }
4	(5)	--Q	4 DN	{ Preparing for MTT }
1,4	(6)	-P	1,5 MTT	
7	(7)	R	A	{ Right disjunct }
7	(8)	--R	7 DN	
2,7	(9)	-P	2,8 MTT	
1,2,3	(10)	-P	3, 4,6, 7,9 vE	
1,2	(11)	(Q v R) → -P	3,10 CP.	

The LEMMON-AID sheets E3, S3, X3, E4, S4 and X4 contain some 48 further sequents whose proofs involve the negation and disjunction rules. These are highly recommended for practice especially in using RAA and vE.

SHEET 5

We now have all the rules for Lemmon's system for the connectives &, →, -, v. Lemmon goes on to introduce a generally useful method of shortening proofs. He gives rules of Theorem Introduction and Sequent Introduction (TI and SI). TI allows us, whenever we have proved a theorem, to give it a name and then to introduce it as a line of proof at any time. It rests on no assumptions at all, and the annotation is 'TI' with the name of the theorem. The proof of the theorem could in principle be interposed, making TI redundant, but this would evidently be a waste of effort and space: since the proof worked last time, it would work this time. I allow, though Lemmon does not, any substitution instance of a proved theorem to be introduced under its name by TI. Notice that TI is not a formal rule because its operation depends on the historical accident of whether a particular result has already been proved.

The rule SI is more general, and more useful. Recall that a provable sequent is supposed to correspond to a provably valid argument form. Given that all the premisses have been validly derived from some assumptions, therefore, its conclusion also follows validly from the same assumptions. (Read that again to make sure you see the point.) Given a proved sequent

$$A_1 \ldots\ldots A_n \;:\; B$$

and a proof in which the formulas $A_1 \ldots\ldots A_n$ all occur on lines, we may proceed to write B as the formula on a new line, resting on the assumptions needed to get $A_1 \ldots\ldots A_n$. In the formal notation for stating rules, SI is:

$$\frac{A_1 \ldots A_n : B \qquad X_1 : A_1 \quad \ldots\ldots \quad X_n : A_n}{X_1 \ldots\ldots X_n \;:\; B}$$

The annotation consists of the n lines on which the various A_i occured and the expression 'SI' with the name given to the sequent $A_1 \ldots A_n : B$. In practice, n tends to be small - usually either n=1 or n=2. By allowing the case n=0 we could see TI as a special instance of SI. What SI gives us, in effect, is a new rule of the system corresponding to each proved sequent. Where we have proved

$$A_1 \ldots\ldots A_n \;:\; B$$

SI gives us the corresponding rule

$$\frac{A_1 \quad \ldots\ldots\ldots \quad A_n}{B}$$

This is a secondary rule, as opposed to the primitive rules &I, vE, etc. A rule is admissible in a given logical system if its addition would not allow anything new to be proved, so that if added it would in principle be redundant. It is a derivable rule if it is admissible not only in the given system but also in any extension of that system with extra rules, connectives, etc. Note that all secondary rules (which Lemmon calls "derived" rules) are derivable and therefore admissible, but that not all admissible rules are derivable.

Lemmon's Chapter 1 and Parts 1 and 2 of Chapter 2 could be read at this point. Note particularly the use he makes of TI and SI. His remarks on the bi-conditional are something of an afterthought and may be ignored for now.

Thus far our formal logic has been a system of <u>proofs</u>. Although the rules (MTT, &E, etc.) were motivated in terms of the conditions in which it would be correct to assert statements of given forms, their formulation and use makes no reference to meanings. <u>The proof system is pure syntax</u>. <u>Questions about truth, falsehood and the like belong to semantics, or the theory of meaning</u>, which gives us another way to investigate validity and invalidity of sequents.

Sheet 4 of these notes, introducing negation, uses simple tables recording the effect of negation on truth and falsehood. The idea of such <u>truth tables</u> extends naturally to other connectives. Every statement has exactly one of two <u>truth values</u> which we shall symbolise 'T' and 'F'. Intuitively, true statements have the value T and false ones the value F. An <u>interpretation</u> of the formal language <u>is simply an assignment of truth values to the atoms.</u> This results in a value for each formula. The rules by which these values are calculated are summarised in the truth tables. For negation, we have

A	-A
T	F
F	T

meaning that for any formula A, when A has the value T -A has the value F and vice versa. For the dyadic connectives, there are four possible cases:

	A	B	A & B	A v B	A → B	A ↔ B
case 1	T	T	T	T	T	T
case 2	T	F	F	T	F	F
case 3	F	T	F	T	T	F
case 4	F	F	F	F	T	T

These tables should be memorised.

It is fairly obvious why the tables for '&' and 'v' are as they are. That for '→', however, is far less clearly correct. The value F in the case where A is true and B false is easily motivated, but why the value T in all the other cases? One answer is given by the (syntactic) provability of the sequents

$$Q \vdash P \rightarrow Q$$

$$-P \vdash P \rightarrow Q.$$

These show that the rules CP, &I, RAA, etc. have the outcome that Q and -P each entail P → Q. So if those rules are correct, Q cannot be true or P false unless P → Q is true. Thus whatever lent plausibility to the rules can now be enlisted in support of the truth table.

Truth tables for complex formulas can be built up from those for their parts. To illustrate, let us compute the value of the formula

(P v Q) & -(Q → -R)

for an interpretation which assigns T to P, T to Q and F to R.

P	Q	R	(P	v	Q)	&	-	(Q	→	-	R)
T	T	F	T	T	T	F	F	T	T	T	F

Having filled in the values of the three atoms P, Q and R we copy those values under the occurrences of the atoms in the formula. Then we fill in the value of each subformula under its main connective, calculating it according to the truth tables. Eventually the value of the whole formula appears under <u>its</u> main connective (in this case the '&'). The process can be repeated for every possible assignment of values to the three atoms:

P	Q	R	(P	v	Q)	&	-	(Q	→	-	R)
T	T	T	T	T	T	T	T	T	F	F	T
T	T	F	T	T	T	F	F	T	T	T	F
T	F	T	T	T	F	F	F	F	T	F	T
T	F	F	T	T	F	F	F	F	T	T	F
F	T	T	F	T	T	T	T	T	F	F	T
F	T	F	F	T	T	F	F	T	T	T	F
F	F	T	F	F	F	F	F	F	T	F	T
F	F	F	F	F	F	F	F	F	T	T	F

In constructing such tables it is easier to compute in columns than in rows. Notice that the construction technique, unlike the process of searching for proofs, is entirely mechanical, requiring no ingenuity or imagination. Truth tables therefore yield an automatic procedure for analysing complex formulas.

Recall that a set X of premisses entails a conclusion A iff it is impossible for everything in X to be true while A is false. Accordingly, we say that a sequent

X : A

is <u>semantically valid</u> iff there is no interpretation giving the value T to all formulas in X and F to A. To <u>test</u> a given sequent for (semantic) validity therefore, we just construct its truth table, consisting of the truth tables of its constituent formulas side by side, and look for a row which has T under the main connective of <u>every</u> premiss and F under the main connective of the conclusion. If there <u>is</u> such a row then the sequent is <u>invalid</u> because there is a way to make the premisses true and the conclusion false. It is clear enough that the truth table test gives a purely mechanical procedure for deciding on the validity or otherwise of sequents. We use a double turnstile

$X \models A$

to record the claim that the sequent X : A is semantically valid.

In practice it is tedious to write out all the rows of a truth table test, and mistakes can easily occur (especially if there are many atoms around). To overcome this problem, we use a quick testing method which only requires us to construct those rows likely to yield a refutation. To illustrate, consider

(P → (Q v R)) → (-(S & Q) → ((P & S) → R)).

We are to test this formula to see if it is a tautology. That is, does it always get the value T regardless of the values of the atoms? Its full truth table has 16 lines and so is laborious to construct. Instead, we note that we are seeking a line giving F under the main connective. The formula is a conditional, and so can get F only if its antecedent gets T and its consequent F. The consequent is also a conditional, so in any refuting assignment <u>its</u> antecedent and consequent must get T and F respectively. Applying this reasoning as far as it will go, we reach

(P	→	(Q	v	R))	→	(-	(S	&	Q)	→	((P	&	S)	→	R))
	T				F	T				F		T		F	F
	2				1	4				3		6		5	7

The numbers show the order in which the values were inserted. Now by applying similar reasonings, such as that if A & B gets T so do A and B, we reach

```
(P → (Q v R)) → (- (S & Q) → ((P & S) → R))
   T          F  T     F    F   T  T  T  F  F
   2          1  4     10   3   8  6  9  5  7
```

Now any subformula must have the same value every time it occurs, and a true conditional with a true antecedent must have a true consequent, so

```
(P → (Q v R)) → (- (S & Q) → ((P & S) → R))
 T  T    T  F   F  T  T  F   F   T  T  T  F  F
 11 2    14 12  1  4  13 10  3   8  6  9  5  7
```

But now Q must get the value T, to make the disjunction Q v R true, and it must also get the value F, to make the conjunction S & Q false. This is plainly impossible, so our original supposition that there could be a falsifying assignment for the whole formula has been refuted.

The general principle of this kind of test is to start by writing T under the main connective of each premiss (if there are any) and F under the main connective of the conclusion. That is, we suppose there were a refuting assignment. Then we reason, using the truth tables, about how such values could possibly have got there. This forces more values, on smaller and smaller subformulas. <u>If an absurdity results</u> - for instance, if some subformula gets forced to have both values - <u>then the sequent we are testing has been shown valid</u>. <u>If a complete assignment of values, free from absurdity, is found then the sequent has been shown invalid.</u>

It sometimes happens that no more assignments of values are forced by what we have, even though the line is incomplete. When this occurs we split the attempted assignment, trying two lines which differ at some chosen point. For validity of what is being tested, <u>both</u> attempted lines must turn out absurd (explain to yourself why). For an example, consider

```
P v Q,  P ↔ Q   ⊨   P & Q
  T       T           F
  1       2           3
```

Here no further assignments are forced by the tables. So we pick a subformula (generally it pays to pick the longest one available, but in the present case we have to pick an atom because there is nothing better around). We pick P. Now set up <u>two</u> attempts, differing in the value assigned to P.

```
P v Q,  P ↔ Q   ⊨   P & Q
  T     T T T       T F F
  1     4 2 5       6 3 7

P v Q,  P ↔ Q   ⊨   P & Q
F T T   F T F         F
6 1 7   4 2 5         3
```

Follow the numbers, and check that you see all the reasoning involved. In each case, though for different reasons, Q has been forced to take two values. Hence either way the supposition is reduced to absurdity, so there can be no refuting assignment, so the sequent is valid.

SHEET 6

Paradoxes of Material Implication

Among the sequents provable in orthodox logic, of which Lemmon's system is one formalisation, are some which must strike the reflective observer as somewhat bizarre or even downright invalid. They include such specimens as:

$P \vdash Q \rightarrow P$

$-P \vdash P \rightarrow Q$

$P, -P \vdash Q$

$P \vdash (Q \rightarrow R) \vee (R \rightarrow S).$

Try filling in actual sentences for the letters in these. For instance:

The People's Flag is Deepest Red;

therefore

either if you're gullible then you'll believe Lemmon,
or if you'll believe Lemmon then you'll believe anything.

This can hardly be claimed as an argument which commends itself intuitively as valid. For one thing, even if the conclusion were true, the premiss would be entirely irrelevant to it. Yet the sequent is valid on a truth table test, and it has a proof in Lemmon's logic. There is thus an apparent mismatch between the orthodox logical properties of the formal connectives and the intuitive data about validity which the system was designed to capture. The rather startling sequents above, and others like them, are known as the paradoxes of material implication.

The philosopher R.J. Fogelin gave an imprecise definition of a paradox of material implication as a provable sequent whose proof requires "funny business". The utility of such a definition, of course, depends on what sense can be given to the notion of "funny business". Let us look at a proof.

1	(1)	P	A
2	(2)	Q	A
1,2	(3)	$P \& Q$	1,2 &I
1,2	(4)	P	3 &E
1	(5)	$Q \rightarrow P$	2,4 CP.

This is a legitimate proof according to the formal rules. The funny business surrounds the claim at line (4) that P has been derived from Q so that CP can apply at line (5). The assumption of Q (assumption 2) is used in the derivation of P alright, but it is intuitively clear, if a little hard to express formally, that it is not used in a way that makes P really depend on it. To say that P came from Q under other assumptions stretches the sense of 'from'. Interpreting a concept in such a way that its sense is stretched like this is what Fogelin means by "funny business". We might suggest a precise definition of 'paradox of material implication' for the fragment of the language involving only the connectives $\rightarrow$ and $-$. A sequent in which no connectives other than $\rightarrow$ and $-$ occur is a paradox of material implication if and only if (a) it is provable, and (b) every proof of it involves either &I or vE.

It is sometimes said that the paradoxes are not really problematic because they just show that the English locution 'IF...THEN' does not mean the same as →. This response will not do, however, since the rules of Lemmon's system are all well motivated by appeal to the inferential properties of the corresponding connectives in natural languages such as English. Since those rules lead to the paradoxes, either the paradoxes are acceptable or the rationale for the rules is not. The above proof uses only the rules A, &I, &E and CP. Anyone who thinks that paradox is not valid for English 'IF' and 'AND' must say which of the rules just mentioned is invalid for the English version and why. So the paradoxes are genuinely paradoxical: they are quite counter-intuitive, yet they follow from convincing principles culled from our understanding of ordinary reasoning. To resolve the situation, something has to give.

The standard, classical response is to accept the paradoxes as valid principles governing natural connectives and learn to love them. Support for this option comes from proofs like the one on the last page which derive the disputed principles from ones it is relatively hard to reject. Often the classical theory is presented as the lesser of two outrages; it is claimed that to abandon &I or CP or something would be much more counter-intuitive than the paradoxes themselves. How cogent such a thought is depends both on intuitions about relative outrageousness and on how great a revision of logic is in fact forced by avoiding the paradoxes. The latter question is a delicate one, and is so far unresolved.

Obviously, the major problem for the classical view is to account for the fact that most people feel a natural resistance to the paradoxes on first encounter. To explain why the paradoxes are repugnant, it is normal to turn to the distinction between appropriateness and truth. There are many ways in which an utterance might be inappropriate even if true. It may be impolite, for instance. Most of these are not interesting for logic, but some are. Importantly, it is in general inappropriate to tell only part of the truth in a long tale if one is in a position to tell all of it more shortly. Thus if you ask me how Hearts did last Saturday and I reply, knowing that they won, 'Well, they didn't lose' then what I say is perfectly true, because winning entails not losing, but by saying it I mislead you into thinking that perhaps they drew, or that I don't know whether they won or drew. So we have two statements, W ('Hearts won') and L ('Hearts lost') such that W entails -L although the utterance of W is appropriate and that of -L is inappropriate. In the same way, it is held, if I know that Q then I mislead by asserting $P \to Q$, for I convey the false impression that I don't know whether Q, or indeed whether P, but only some connection between them. This does not mean that Q doesn't entail $P \to Q$, for logic is only concerned with preservation of truth, not with any other impressions our premisses and conclusions may convey. Reflection on the difference between appropriateness and truth has helped to convince many philosophers that the classical response to the paradoxes faces no serious difficulties. The "inappropriateness" argument seems rather weak to me, however, for it fails to explain why we feel less resistance to the inference from P to $P \vee Q$ than to that from Q to $P \to Q$.

The alternative to giving up our intuition of paradoxicality is of course to reject the paradoxes as invalid and to amend our formal logic so as to avoid being able to prove them. This option has generally been less popular than the classical one, though it has gained ground over the last 25 or 30 years, since respectable alternative logics became available. The easiest way into such an alternative system is to pay more attention than Lemmon does to the ways in which premisses can be combined with each other in sequents. In the proof cited earlier, two assumptions were put together by &I and then put asunder by CP. Neither move is in itself objectionable: the funny business lies in combining the two. What we need is a distinction between the way in which &I combines assumptions and the way in which CP requires them to have been combined. Such a distinction is aimed at matching the different ways in which premisses can be used to generate a conclusion.

To produce a paradox-free logic in place of Lemmon's we give ourselves two notations for combining assumption numbers on the left. Let us use the comma (,) and the semicolon (;) for these. Then & is tied to , as in the usual presentation, whereas $\rightarrow$ is tied to ; . So we now have the rules

$$\frac{X \ : \ A \qquad\qquad Y \ : \ B}{X, Y \ : \ A \ \& \ B} \qquad \&I \qquad \text{(just as before)}$$

but

$$\frac{X \ : \ A \rightarrow B \qquad\qquad Y \ : \ A}{X; Y \ : \ B} \qquad MPP \qquad \text{(note the semicolon)}$$

$$\frac{X; A \ : \ B}{X \ : \ A \rightarrow B} \qquad CP \qquad \text{(and again)}$$

The rules MTT, RAA and vE will also have to be modified slightly. Notice how the derivation of the paradox given on page 1 above is now blocked. It goes through as far as line (4)

1,2 (4) P

but to apply CP in its new form we should require the stronger

1;2 (4) P

which is not forthcoming. It would require a way of involving Q in the derivation of P more deeply than just by its coming in and going out again as happens in this proof. This can be shown (though I shall not show it here) to be impossible in the amended system. The other paradoxes are likewise, and for roughly similar reasons, underivable.

It is a matter of unsettled philosophical debate whether such a restricted system, a so-called "relevant" logic, captures the intuitions better or worse than the standard brand. It is also an open question whether the restricted logic is adequate to allow important theories, in mathematics for instance, to be reconstructed formally. The two formal systems, Lemmon's and the relevant logic, give different and conflicting accounts of valid reasoning. Part of the interest of formal logic is that it both generates philosophical problems such as that of choice between these accounts and provides the tools with which these problems may be tackled. Hard questions arising include:

Are the paradoxes useful? Are they needed? If so, for what?

What difference does it make (e.g. to mathematics) if we manage without them?

What, exactly and in full generality, <u>is</u> a paradox of material implication?

What intuitions <u>are</u> there in this area? How may they change as a result of logical investigations such as we are now undertaking?

What considerations bear on the question of which logic (if either) is <u>correct</u>?

Soundness and Completeness

We have developed propositional logic in two ways: syntactically via a proof system and semantically via truth tables. These two developments give us two relations of logical consequence or validity. The next question is whether the two are in fact one, whether the same sequents are valid semantically as are provable from the rules. The answer is "yes". This answer comes in two parts. Firstly, every provable sequent is valid according to truth tables. This is called the soundness of Lemmon's rules. It amounts to the correctness of those rules (MPP, vE, etc.) for logic as semantically given. Secondly and conversely, every (semantically) valid sequent is also derivable in the proof system. This is the completeness of the rules, for it amounts to their having omitted nothing valid.

Soundness is proved rather laboriously by showing two facts. The first is that every sequent introduced into a proof by the rule of assumptions is valid. This is trivial (explain to yourself why). The second is that every use of one of the other rules preserves validity in the sense that if its inputs are valid then so is its output. This has to be shown by cases and involves a lot of detailed reference to the truth tables. Lemmon gives the argument on pages 77 - 82 of his book. The upshot is that there is no way for any invalid sequent to get into a proof, so every provable sequent is valid.

Completeness is much harder to prove. There is a completeness theorem of sorts in Chapter 2 of Beginning Logic, but it leaves out the difficult case in which a valid sequent has infinitely many premisses. To deal with such deep difficulties we need much heavier mathematics than can be covered in this course - the full completeness theorem is proved and discussed in Logic 2. For the moment my confident assertion will have to serve (Proof by Authority).

There are several reasons for showing soundness and completeness.

(a) Since the two notions of entailment coincide, any rationale supporting one can be enlisted in support of the other. Thus on the one hand the truth table for 'v' which is obviously correct can be used to justify the rule vE which might otherwise be rather obscure, while on the other hand the truth table for '→' which is somewhat under-motivated is supported by whatever lent credibility to CP, &I, RAA and the like.

(b) Truth tables give an easy decision procedure for validity of sequents. This transfers via the completeness theorem to a decision procedure for provability, giving us a way of checking whether sequents are derivable or not.

(c) In particular, soundness assures us that the proof system is in several senses consistent. Minimally, not everything can be proved in it. More strongly, there are formulas which do not entail contradictions according to it. These things we expect of any well-motivated logic, of course, but it is welcome that they can be proved to hold.

(d) A truth table test, especially one using a slick method such as a one-line test or a truth tree, will often give not only the information that a sequent is valid but also some reasons why it is valid. We find out what prevents it from having an invalidating assignment, and by studying this we can often see how the sequent might be proved. Proofs often simulate truth-table reasoning in a way which must be picked up from examples.

SHEET 7

We now turn from the logic of sentential connectives to a more comprehensive system allowing us also to take account of logical inferences which turn on the behaviour of words expressing generality. Such locutions as 'all', 'some' and 'none' come within its scope. We shall now be concerned with the internal structure of atoms (i.e. of formulas without connectives) as well as with compounds built out of them. Consider the following, for example:

All logicians are admirable.
Some philosophers are not admirable.
Therefore not all philosophers are logicians.

This is plainly valid in virtue of its form. To demonstrate and explain its validity, however, we need a richer language and more formal machinery than are available in the logic we have treated so far.

First we need to be able to name things. In logic, a <u>proper name</u> is a simple (that is, unalalysable) linguistic particle serving to pick out and refer to an individual thing. Examples of ordinary proper names include

Scotland
Thirty-four
Hearts
Margaret Thatcher

These designate a country, a number, a football club and a human being, I suppose. Lemmon uses lower case letters as names (he calls them "terms"):

m, n, o, ...	proper names
a, b, c, ...	arbitrary names.

The distinction between <u>proper</u> and <u>arbitrary</u> names does not apply at the level of pure logic, so I shall simply call them all names or terms. Logically proper names such as are symbolised by these letters differ from names of ordinary language in that they are guaranteed to pick out exactly one individual. Ordinary names like 'Mary' designate ambiguously (there are many people called Mary), or they may not designate at all (consider 'El Dorado' or 'Zeus', for instance).

So we can name things. Next we need to be able to describe them. For this we use <u>predicates</u>. A predicate is an expression which yields a sentence when appropriately many names are inserted in it. For examples, consider:

is larger than Andorra
is a multiple of ten
can beat
supports

The first two of these require one name to make a sentence; the last two need two names each. Lemmon uses upper case letters

F, G, H, ... ,

which I shall call <u>predicate symbols</u> or <u>predicate letters</u>, to stand in for predicates as these occur in formulas. Within each piece of discourse, each predicate is specified as being one-place, two-place, etc. according to the number of names it takes. This fixed "adicity" is another difference from (arity?) ordinary language, where predicates like 'are related' take no fixed number of names.

With this apparatus we can already form subject-predicate sentences like 'Edinburgh is a city' and 'John loves Mary'. These can be formalised 'Ce' and Ljm respectively (with obvious mnemonic choice of letters). Note that predicates precede names. Using connectives, we can now say

John and Mary love each other	Ljm & Lmj
John and Mary love themselves	Ljj & Lmm
Mary's love for John is not reciprocated	Lmj & -Ljm.

So far, however, we cannot express generality. We cannot say

Mary loves everybody
Somebody loves Edinburgh

and the like. For cases like this we introduce <u>variables</u> and <u>quantifiers</u>. In order to express a universal claim like

Everything fades

we first paraphrase it to give the slightly more awkward

Take anything you like; it fades.

To render this expressible in the formal language, using F for 'fades':

Take any thing; call it x: Fx.

or

For each and every thing, x, Fx.

Here the letter x is not a name, although it goes in the place of one, for it is not designating any particular thing. Rather, it is a variable. The locution 'for every thing x' is symbolised by putting the x in parentheses, so our formal notation for 'Everything fades' is now

(x)Fx.

The expression '(x)' is a universal quantifier. It must be followed by what would be a formula if the variable x were replaced by a name. There is a separate universal quantifier for each variable x, y, z, x', y', etc. Now consider how to express in the same notation such a statement as

All philosophers are demented.

Take any philosopher, x: x is demented.

Take any thing, x: if x is a philosopher, x is demented.

(x)(Fx → Dx). [using 'F' for 'Filosopher']

Notice that in the formalisation of such statements the universal quantifier naturally goes in tandem with the conditional, →.

It is also very useful to have an expression corresponding to the claim that <u>some</u> object of a given kind exists or has a given property. To formalise

Flying saucers exist

we turn it into the equivalent

There exists at least one thing which is a flying saucer.

or, with a variable,

There is some thing, x, such that x is a flying saucer.

The locution 'There is some thing, x, such that.....' is symbolised (∃x). This is an <u>existential quantifier</u>. So, using F for 'is a flying saucer':

(∃x)Fx.

Note that at this stage we do not distinguish between the singular 'There is <u>a</u> flying saucer' and the plural 'There <u>are</u> flying saucers'.

To express 'Some F is G' or 'Some Fs are Gs' we again combine the quantifier with a connective. Consider

> Some logicians are cuddly.
> There is [such a thing as] a cuddly logician.
> There is some thing, x, such that x is both a logician and cuddly.
> (∃x)(Lx & Cx).

In the expression of such existential claims the natural connective to use is conjunction (&). DO NOT ATTEMPT TO USE → FOR THIS PURPOSE.

The definition of a <u>formula</u> of first order or quantificational logic is rather more complicated than that given earlier for the propositional calculus, since the language is so much richer. The alphabet is extended to give us

terms (i.e. names)	a, b, c, d, m, n, o, a', b', ...
variables	x, y, z, x', y', z', x'', y'', ...
predicate symbols	F, G, H, F', G', ...

and the reverse "E". An <u>atom</u> is now either a propositional atom or a predicate symbol followed by one or more terms. The clauses for constructing formulas are:

1. An atom is a formula.

2. Where A is a formula, so is -A. (i.e. '-' followed by A)

3. Where A and B are formulas, and δ is a dyadic connective, (A δ B) is a formula.

4. Where A[t] is a formula containing term t and v is a variable not occurring in A[t], and A[v] is the result of substituting v for t throughout A[t], both (v)A[v] and (∃v)A[v] are formulas.

It is worth noting some consequences of clause 4 as given here and by Lemmon. Here are some formulas and some non-formulas.

Formulas	Not formulas
Faa	Fxx
Fba	Fxy
(∃x)Fxa	(∃x)Faa
(∃x)Fxx	(∃x)Fxy
(y)(∃x)Fxy	(x)(∃x)Fxx

That is, our formal language has no place for "free variables" not bound by quantifiers, or for "vacuous quantifiers" which fail to bind variables, or for "confusion of bound variables" whereby one quantifier occurs inside the scope of another binding the same variable.

The problem of translating English sentences into formal notation is perhaps as hard as any technical one raised by the logic in this course. The reason is that there are no infallible rules or algorithms to do the job for us. All we can do is to paraphrase, using our command both of the formal system and of the natural language, and relying on imagination and inventiveness.

We know how to say 'Some Footballers are Hairy' and 'All goats are Hairy':

$(\exists x)(Fx \ \& \ Hx)$ $\qquad$ $(x)(Gx \rightarrow Hx)$.

Now how about 'No goats play football'? Well,

$-(\exists x)(Fx \ \& \ Gx)$ $\qquad$ or equivalently $\qquad$ $(x)(Gx \rightarrow -Fx)$.

Examples like this can easily become very intricate, especially if we add a two-place predicate (i.e. a relation) such as '...kicks...'. A sentence like 'Only hairy footballers kick goats' is <u>ambiguous</u>. That is, it sustains two quite different readings. It might mean either of

(1) Whoever [or whatever] kicks a goat is a hairy footballer (i.e. all episodes of goat-kicking are by hairy footballers).

(2) Any footballer who kicks a goat is hairy.

'Aristotle kicks a goat' means that there is some goat Aristotle kicks:

$(\exists y)(Gy \ \& \ Kay)$.

So the two readings above can be rendered:

(1) $(x)((\exists y)(Gy \ \& \ Kxy) \rightarrow (Fx \ \& \ Hx))$

(2) $(x)((Fx \ \& \ (\exists y)(Gy \ \& \ Kxy)) \rightarrow Hx)$.

Try to see why these are indeed formalisations of the two sentences, and why they do not mean the same. The exposure and exact explanation of ambiguity is one of the important applications of formal logic. It can often help to clarify difficult issues in philosophy, mathematics, linguistics and even real life! Arguments like

All goats are hairy
Therefore any footballer kicked by a goat is kicked by something hairy

can be formalised, given a little thought:

$$\frac{(x)(Gx \rightarrow Hx)}{(x)\ (\ [Fx \ \& \ (\exists y)(Gy \ \& \ Kyx)] \rightarrow (\exists y)(Hy \ \& \ Kyx)\ )}$$

and will turn out provable in the logical system to be developed in the next part of the course. That such complex argument forms can be proved valid in modern logic is a major respect in which it is an advance on the kind of logic in use prior to this century.

One particular potential ambiguity in English gives rise to a common confusion which is important enough to have been given a name. This is the "quantifier shift" and gives rise to the "quantifier shift fallacy". Consider

Everyone kicks someone.

This could refer to one unfortunate universal kickee, or it might allow different kickers different targets. That is, it could translate as either of

$(x)(\exists y)Kxy$

$(\exists y)(x)Kxy$.

These are obviously not equivalent. Yet in more complex cases it is rather easy to get them confused. The "fallacy" is to infer the second from the first. You, of course, would never do such a thing.

SHEET 8

This sheet concerns the proof system of <u>first-order logic</u> or the <u>lower predicate calculus</u>. The notion of "proof" is much as it was for sentential logic, except that we have a new definition of "formula" (see Course Notes 7) and some new rules for introducing and eliminating quantifiers. The rules for connectives are simply taken over from the sentential case.

The rules of UE (Universal Elimination) and EI (Existential Introduction) are very easy to state, motivate and use. <u>UE corresponds to the valid argument form known as "instantiation":</u>

Everything is F   Therefore a is F.

Since a is one of the things, what is true of everything must be true of a in particular. Formally, then, where (v)A[v] is a universally quantified formula and A[t] results from it by dropping the quantifier and substituting term t for variable v throughout,

$$\frac{(v)A[v]}{A[t]}$$

or in unabbreviated format

$$\frac{X \;:\; (v)A[v]}{X \;:\; A[t]}$$

The annotation consists of just the one line number and the expression 'UE'. For an example, consider the following very simple proof.

$(x)(Fx \rightarrow Gx),\ (y)Fy \ \vdash\ Gm$

1	(1)	$(x)(Fx \rightarrow Gx)$	A
2	(2)	$(y)Fy$	A
1	(3)	$Fm \rightarrow Gm$	1 UE
2	(4)	Fm	2 UE
1,2	(5)	Gm	3,4 MPP.

The rule EI is equally easy. It corresponds to the obviously valid form

a is F   Therefore something is F.

Formally, where (∃v)A[v] is an existentially quantified formula, and A[t] results by deleting the quantifier and substituting t for v as before,

$$\frac{A[t]}{(\exists v)A[v].}$$

The annotation is the one input line number and 'EI'. Note that neither EI nor UE has any effect on the set of assumptions. Note also that the formulation of EI allows that not all occurrences of t in A[t] need turn into occurrences of v. For example, suppose that t is the name b and that $(\exists v)A[v]$ is the formula

$(\exists x)(Fx \rightarrow Fb)$

then A[t] is the formula

$Fb \rightarrow Fb$.

This allows the following proof.

$\vdash (\exists x)(Fx \rightarrow Fb)$

1	(1)	Fb	A
	(2)	$Fb \rightarrow Fb$	1,1 CP
	(3)	$(\exists x)(Fx \rightarrow Fb)$	2 EI.

The rules to introduce the universal quantifier and to eliminate the existential one are slightly harder to state and use because they are subject to some restrictions. It is convenient to approach them by comparing the quantifiers with & and v. The universal quantifier behaves rather like conjunction. To say 'All politicians are devious' is like saying 'Thatcher is devious <u>and</u> Kinnock is devious <u>and</u> Steele is devious <u>and</u> ....' Similarly, the existential quantifier is a kind of disjoiner: <u>someone</u> finds vE exciting iff <u>either</u> Socrates <u>or</u> Zico <u>or</u> Junior <u>or</u> .... finds vE exciting. Listing all the objects in the universe in this way is not in general practicable, however, which is why we have quantifiers.

The rule UE corresponds closely to &E. &E takes us from a conjunction to one of its conjuncts. UE takes us from a generalisation (like a big conjunction) to one of its instances. Similarly, where vI gives us a disjunction based on one of its disjuncts, EI gives us an existential claim based on one of its instances. The remaining quantifier rules correspond similarly to &I and vE.

&I takes us from <u>both</u> conjuncts to a conjunction, so UI ought to take us from <u>all</u> the instances to a generalisation. Unfortunately, the universe might be infinite; and even if it is not, some things might have no names. Physicists do not have to <u>name</u> each individual electron before they can reason about them, for instance. So it is usually impossible to give all instances of a generalisation. What we do, therefore, is to pick a <u>typical</u> one and reason from it. What is true of a truly typical individual is true universally. 'Let ABC be a triangle', we say, and go on to infer that at most one of its angles is obtuse or whatever. Provided no assumption was made about ABC apart from its triangularity, the result stands proved for <u>all</u> triangles. In principle the proof could be repeated for <u>any</u> particular triangle we cared to pick. The formal rule corresponding to this kind of reasoning is as follows. Let A[t] be a formula containing term t, and let v be a variable not occurring in it. Then let $(v)A[v]$ be the result of substituting v for t throughout A[t] and prefixing the universal quantifier. The rule is then

$$\frac{X \ : \ A[t]}{X \ : \ (v)A[v]}$$

provided t does not occur in any of the formulas in the set X.

The proviso is to ensure that the term t is indeed typical. It does this by ruling out any assumptions which might constitute special pleading about the object picked out by t. Notice that the direction of substitution this time is v for t, not t for v as it was for UE and EI. This rules out the fallacious "proof"

1	(1)	Fb	A
	(2)	Fb → Fb	1,1 CP
	(3)	(x)(Fx → Fb)	2 UI ?????

Of course, it would have been entirely in order to infer

(3) (x)(Fx → Fx) 2 UI.

Here is another proof, showing UE and UI working together.

(x)(Fx → Gx) ⊢ (y)(-Gy → -Fy)

1	(1)	(x)(Fx → Gx)	A
2	(2)	-Ga	A
1	(3)	Fa → Ga	1 UE
1,2	(4)	-Fa	2,3 MTT
1	(5)	-Ga → -Fa	2,4 CP
1	(6)	(y)(-Gy → -Fy)	5 UI.

The proof strategy here is worth a second look. The desired conclusion is universal in form, so it will be derived from a typical instance by UI. That is, the intermediate goal is to prove the sequent

(x)(Fx → Gx) ⊢ -Ga → -Fa.

The new conclusion is a conditional, so we assume its antecedent for CP. We are now aiming for

(x)(Fx → Gx), -Ga ⊢ -Fa

which we achieve at line (4) by an easy UE-MTT combination. The assumption involving the name, a, upon which we wish to generalise must be discharged before we do so in order that the constraint on UI be met.

EE is related to vE as UI is to &I. The upshot of vE is that a disjunction entails just what follows regardless of which disjunct we pick. EE reflects the analogous principle that if a conclusion B follows from A[t] <u>whatever</u> t might be, then B follows from the existence of <u>something or other</u> satisfying the description A. As before, we use a <u>typical</u> term t, so that the argument to B in no way depends on what term it is. To guarantee this, t must not occur in any auxiliary assumptions used in deriving B, and must not occur in B itself. Let A[t] and v be as above (in the case of UI) where the variable v is substituted for term t. Then

$$\frac{X \;:\; (\exists v)A[v] \qquad Y,\, A[t] \;:\; B}{X,\, Y \;:\; B}$$

provided t is not in B or in any formula in Y.

The annotation consists of the two input line numbers (of the existential and of B) together with the discharged assumption number (of the typical instance A[t]) and the expression 'EE'.

Applications of EE have to be "set up" in a way reminiscent of vE (though not as complicated). Given an existential formula to be broken up by EE, first assume an instance of it - what Lemmon calls a "typical disjunct". Then derive the conclusion from it. To ensure that the term used is really typical, choose a name which is not in any other assumption to be used and which is not in the desired conclusion. Notice that the assumption A[t] is just that - a result of the rule of assumptions - and not <u>in any way</u> derived from (∃v)A[v]. Notice too that the rule has only two input lines although three numbers get cited in the annotation. The following proofs illustrate EE and the other quantifier rules.

1. (∃x)Fx ⊢ (∃y)Fy

1	(1)	(∃x)Fx	A
2	(2)	Fa	A
2	(3)	(∃y)Fy	2 EI
1	(4)	(∃y)Fy	1,2,3 EE.

Here line (2) is the assumption of a typical instance of line (1). The two input lines for line (4) are thus (1) and (3). Note that EI applies before EE in order to remove the term t (in this case the 'a') from the conclusion.

2. (∃x)(Fx & Gx), (x)(Gx → Hx) ⊢ -(x)(Fx → -Hx)

1	(1)	(∃x)(Fx & Gx)	A	
2	(2)	(x)(Gx → Hx)	A	
3	(3)	(x)(Fx → -Hx)	A	{ for RAA purposes }
4	(4)	Fa & Ga	A	{ typical case of (1) }
2	(5)	Ga → Ha	2 UE	
3	(6)	Fa → -Ha	3 UE	
4	(7)	Fa	4 &E	
4	(8)	Ga	4 &E	
2,4	(9)	Ha	5,8 MPP	
3,4	(10)	-Ha	6,7 MPP	
2,3,4	(11)	Ha & -Ha	9,10 &I	
2,4	(12)	-(x)(Fx → -Hx)	3,11 RAA	
1,2	(13)	-(x)(Fx → -Hx)	1,4,12 EE.	

Note that we cannot apply EE before RAA here, since 'a' occurs in 'Ha & -Ha'.

3. -(x)(Fx → Gx), (x)(Fx → Hx) ⊢ -(x)(Hx → Gx)

1	(1)	-(x)(Fx → Gx)	A
2	(2)	(x)(Fx → Hx)	A
3	(3)	(x)(Hx → Gx)	A
4	(4)	Fa	A
2	(5)	Fa → Ha	2 UE
2,4	(6)	Ha	4,5 MPP
3	(7)	Ha → Ga)	3 UE
2,3,4	(8)	Ga	6,7 MPP
2,3	(9)	Fa → Ga	4,8 CP
2,3	(10)	(x)(Fx → Gx)	9 UI
1,2,3	(11)	(x)(Fx → Gx) & -(x)(Fx → Gx)	1,10 &I
1,2	(12)	-(x)(Hx → Gx)	3,11 RAA.

It is important to see that we CANNOT apply UE to line (1), because it is of the form -A, not of the form (v)A[v]. Rules only apply to MAIN operators.

SECTION 1. ON NONEXISTENCE

Many theorems of ordinary predicate logic have as main operator an existential quantifier. For example,

$\vdash \quad (\exists x)(Fx \rightarrow Fx)$

1	(1)	Fa	A
	(2)	Fa → Fa	1,1 CP
	(3)	(∃x)(Fx → Fx)	2 EI.

Thus it is provable, as a matter of sheer logic, that something or other exists, that the universe is not empty. Well, it is <u>true</u> that the universe is not empty, and we might look for metaphysical arguments for supposing this truth a <u>necessary</u> truth. What is unsettling about the logical guarantee of existence, however, is that logic only gives <u>bare</u> existence, telling us nothing about <u>what</u> has to exist. Metaphysical reasons why something exists, rather than nothing, tend to be reasons why some <u>kind</u> of thing has to exist (physical objects, space-time points, minds, God, numbers, sets or whatever). Logic will have none of this: provably something exists, but nothing provably exists. This is odd.

Another oddity concerns the <u>existential import</u> of universal quantification. The provable sequent

$(x)Fx \quad \vdash \quad (\exists x)Fx$

attests to the fact that "all implies some", that what is true of everything is, as a matter of pure logic, true of something. That might not be a bad claim for logic to make; the oddity is that there is no parallel claim in cases of restricted quantification over things of a given kind. We cannot validly argue

All men are mortal	(x)(Hx → Mx)
So some men are mortal	(∃x)(Hx & Mx)

because logic (rightly in my view) leaves open the possibility that there be no human beings at all. There is an asymmetry between the logic of 'men' and that of 'things' which is rather unpleasing.

Yet a third cause for concern is orthodox logic's unconvincing treatment of the problem known picturesquely as "Plato's Beard". This is the problem of how we can truly say of something that it does not exist. Pegasus (the mythical flying horse) does not exist, for example, but in order to say so we need to use the name 'Pegasus'. Now if names get their meanings by referring to objects, this particular name must fail to get a meaning, and so it cannot meaningfully be used to say anything at all. The problem generalises to that of giving an account of the meaning and use of non-referring names in all contexts, not just in the context '.... does not exist'. "Logically proper names" are always guaranteed to pick out exactly one thing, so names like 'Pegasus' are taken by orthodox logical theory either to be unformalisable or to be short for descriptions (so 'Pegasus does not exist' really means 'There are no flying horses' or something like that). But there appear to be no grounds other than sheer dogma for holding vacuous names to be unformalisable, while the view that they are translatable as descriptions misses the strong intuition that 'Pegasus does not exist' is <u>about Pegasus</u> (or putatively about Pegasus) in a way that 'Flying horses do not exist' is not about flying horses but about the world, saying of it that none of the horses it contains can fly.

One way of answering the above worries is to change orthodox logic a little, giving what is known as <u>Free Logic</u>. We need to add a special new monadic predicate symbol 'E' for "exists". Then the quantifier rules get changed slightly.

UE. $$\frac{(v)A[v] \qquad Et}{A[t]}$$

EI. $$\frac{A[t] \qquad Et}{(\exists v)A[v]}$$

UI. $$\frac{X, Et : A[t]}{X : (v)A[v]}$$

EE. $$\frac{X : (\exists v)A[v] \qquad Y, Et, A[t] : B}{X, Y : B}$$

The definitions of the formulas (what gets substituted for what) and the restrictions (that t not occur in X, etc.) are exactly as in the orthodox case. Free logic needs the extra input line for UE and EI, assuring that t exists, because it allows for empty names like 'Pegasus'. We do not want to allow the argument

Pegasus is nonexistent
Therefore there <u>exists</u> something nonexistent

as an instance of EI, for example. Nor do we want to argue by UE from

All horses are flightless

to

If Pegasus is a horse then Pegasus is flightless.

In the cases of UI and EE, the discharge of the extra assumption Et amounts to the point that the quantifiers only range over what exists. To be sure that all horses are flightless we need a guarantee that Red Rum is flightless, perhaps, but not one that extends as far as Pegasus. A few examples will be more useful than verbal explanations of the free logical rules.

$\vdash$ (x)Ex

1	(1)	Ea	A	
	(2)	(x)Ex	1,1 UI.	{i.e. from (1) discharging 1}

(x)Fx $\vdash$ (x)(Fx v Gx)

1	(1)	(x)Fx	A	
2	(2)	Ea	A	{ necessary for UE to work }
1,2	(3)	Fa	1,2 UE	
1,2	(4)	Fa v Ga	3 vI	
1	(5)	(x)(Fx v Gx)	2,4 UI.	{ discharging assumption 2 }

(x)(Fx → Gx), (∃x)(Fx & Hx) ⊢ (∃x)(Gx & Hx)

1	(1)	(x)(Fx → Gx)	A	
2	(2)	(∃x)(Fx & Hx)	A	
3	(3)	Fa & Ha	A	{ typical instance for EE }
4	(4)	Ea	A	{ also for discharge by EE }
1,4	(5)	Fa → Ga	1,4 UE	{ note (4) needed }
3	(6)	Fa	3 &E	
1,3,4	(7)	Ga	5,6 MPP	
3	(8)	Ha	3 &E	
1,3,4	(9)	Ga & Ha	7,8 &I	
1,3,4	(10)	(∃x)(Gx & Hx)	4,9 EI	{ note (4) again }
1,2	(11)	(∃x)(Gx & Hx)	2,3,4,10 EE.	

The second of these proofs should be compared with the usual classical proof of the same sequent, which of course does not involve assumption 4. Perhaps more important than the question of what ordinary sequents are still provable in free logic is that of which standardly provable ones have no proofs there. The following are unprovable:

: Ea
: (∃x)Ex
: (∃x)(Fx → Fx)
(x)Fx : Fa
Fa : (∃x)Fx
(x)Fx : (∃x)Fx

The unprovability of these in free logic is an outcome of two things: that free logic allows for terms which fail to refer and that it allows the empty universe. If nothing at all exists, then, for any predicate F, (x)Fx is true and (∃x)Fx is false.

The motivating worries of a page or two back are now settled, or at least less bothersome than they were. That something exists is no longer provable, so there is no longer a puzzle as to why this should be a logical truth when nothing in particular has to exist. The asymmetry between the existential import of unrestricted quantification and the lack of it where quantifiers are restricted has now vanished, also because the empty universe is tolerated. As for Plato's Beard, well it has been trimmed at least. We can allow empty names into the language without getting any obvious nonsense. We have a perfectly straightforward way of formalising 'Pegasus does not exist' in the formula -Ep. We have not settled the issue of whether to regard the name 'Pegasus' as simply not denoting at all or whether to regard it as denoting a nonexistent object. Free logic is neutral between these two positions.

My own feeling is that although free logic is an advance on the standard system it is not sufficient for a solution of the outstanding problems raised by non-existence. My intuition is that while it is literally true (as a truth about this world) that Pegasus is a mythical horse, and it follows that Pegasus is mythical, it is not literally true, and therefore does not follow, that Pegasus is a horse. That is, a mythical flying horse is a kind of mythical horse, and a mythical horse is a kind of mythical thing, a mythical horse is <u>not</u> a kind of <u>horse</u>. Such an intuition cannot yet be accomodated in free logic as formulated above, so it seems there is much work still to be done.

Note finally that it is possible to adopt an eclectic attitude to free logic and the usual system, using standard logic for its simplicity in situations where the empty universe and vacuous names are not countenanced but keeping free logic in reserve for use where these things become important.

(lecture 22)

SECTION 2 VAGUENESS

Serious problems are for logical theory are raised by the fact that most of the descriptive expressions of natural languages are vague. That is, there is usually a more or less imprecise borderline between those cases to which a given description applies and those to which it does not. A couple of examples will make the problem clear.

First, consider a long series of coloured patches shading very gradually from red, through orange, to yellow. We can imagine the difference between each and its neighbour to be so slight as to be imperceptible. Suppose there are 10,000 patches in the series. Now patch 1 is red. Moreover, of any two patches which are indiscriminable in respect of colour, if one is red then so is the other. By repeated applications of MPP, then, we are led to the conclusion that patch 10,000 is red. This conclusion is false, because patch 10,000 is yellow, not red. There must be something wrong with the argument.

Second, consider the chickens and eggs. Suppose Darwin was right about evolution: that it happens not in sudden great mutations but by a gradual accumulation of small ones. Any animal and its immediate offspring then belong to the same species according to the usual tests: they have similar structure; they are genetically very similar indeed; they are cross-fertile. So chickens can come only from chicken eggs, which in turn can be laid only by chickens. Consequently there have always been chickens, even 300 million years ago, before birds evolved.

The problem in each of these cases is that the relevant predicate ('....is red' or '....is a chook') is insensitive to sufficiently small changes in a respect (shade, genetic composition), but is sensitive to sufficiently large changes in the same respect. Yet the large changes are made up of small ones. The logical argument involved is extremely simple, consisting of a large number of conditional premisses

If patch 1 is red then patch 2 is red
If patch 2 is red then patch 3 is red
etc.

together with an initial antecedent

Patch 1 is red

from which the false conclusion follows by a mere 9999 uses of MPP. In an alternative form of the argument, the conditionals could all be obtained by UE from the generalisation

For any number n, if patch n is red then patch n+1 is red.

There are only three possible ways out of the difficulty. These are:

(a) Deny that the problem is well posed. That is, hold either that no logic is applicable to vague expressions or that such logic as does apply is not formalisable.

(b) Maintain that the argument is invalid. That is, hold that MPP in the form used above is not a valid principle but a fallacy.

(c) Hold that although the argument is legitimately set up and valid it is not damaging because one of its premisses is false. That is, claim that there is a last red patch in the colour series and a first chicken in the evolutionary series.

What might look like a fourth option, to accept the conclusion that every coloured object is red, that we are all chickens, etc., really collapses back into option (a) because other rules of our language dictate that the predicate '...is red' is to be witheld from lemons, whence the fourth option involves holding the rules of our language to be inconsistent. But if they are inconsistent then they are not (coherently) formalisable.

Some philosophers try to evade the problem by maintaining not that it is impossible to apply logic to vague discourse but that there is no need to do so. Respectable parts of language, such as natural science, are held to be capable of getting by without vagueness, so logicians are not required to worry about such things. My response to this is threefold. Firstly, it has not been established that much of natural discourse can be freed from imprecision. Secondly, even if, say, science can be done without vagueness, there is no obvious reason why it should be. Thirdly, even if it should, that is no solution to the problem afflicting words like 'red' and 'chicken'. Problems are not solved by changing the subject.

Option (a) is unattractive. It involves holding that there is no coherent way of reasoning in most of ordinary language. This is evidently in conflict with the data - with the fact that we do argue, theorise and criticise using such language, and that we apply criteria of rationality to each other's theorising, etc. in a reasonably stable way. Some formal logicians may be so keen to preserve their abstract theory from confrontation with uncomfortable evidence that they are prepared thus to consign most of what people do when reasoning to the category of the utterly irrational, the uninvestigable. Such an attitude shows a lack of respect for the ordinary. The rules in force governing assertibility in natural languages may not be those of a given formal calculus, or even much like them, but they are rules nonetheless, and can be stated too. There are in any case some formal rules, like &I and &E, which have not been shown to lead to any problems and whose applicability in cases of vagueness seems straightforward. My objection to option (a) is that it retreats too far, securing logic against refutation only by cutting it off from applicability to real life.

Option (c) is initially less repugnant. It corresponds to the natural response that "you have to draw the line somewhere". If one of the conditional premisses is to come out false, there must be some particular number n such that it is false that if patch n is red then patch n+1 is red. In symbols,

$$-(R[n] \rightarrow R[n+1]).$$

By elementary truth functional logic, this is equivalent to

$$R[n] \;\&\; - R[n+1].$$

The problem faced by option (c) is to give an account of how such a thing could be. It seems, after all, that if you are committed to describing something as red then you are committed to describing anything indistinguishable from it as red also. The meaning of "red" is fixed only by the use we make of it, so there would seem to be no way its applicability could turn on distinctions too fine to be drawn. This problem is even plainer in the case of the predicate '....looks red' rather than '....is red'. How could there be two objects such that (i) no difference of colour between them is detectable just by looking, but (ii) one of them looks red and the other does not? The chickens and eggs raise a related form of the same difficulty. There are reasonably well established criteria for sameness of species, though since no two individuals (other than clones) are precisely alike genetically these criteria are subject to certain degrees of imprecision. Nonetheless, any two animals one of which is the parent of the other will fall well within the area where the criteria are satisfied. Yet it is to be held that one of these animals is a chicken and the other is not. Again it is far from obvious that such a description of the situation is coherent.

Option (b) is also not an easy way out. The only rule of inference needed for the problematic derivations is apparently MPP; and surely no rule is more deeply embedded in our natural understanding of connectives. In an important paper on the subject ('Wang's Paradox', reprinted in Truth and Other Enigmas), Michael Dummett puts the point thus: '..... [To abandon MPP] seems a desperate remedy, for the validity of this rule of inference seems absolutely constitutive of the meaning ..... of "if".' The supporter of option (b) must therefore tread a very fine line indeed, providing an alternative logic in which the deduction equivalence and MPP are retained as "constitutive of the meaning of 'if'", while the above derivations which appear to employ MPP are somehow blocked.

I believe that the way to tread that line is, as in the otherwise quite distinct case of relevant logic, to distinguish between two ways of combining premisses or assumptions. As in that case, we could write X , Y for the result of simply collecting up the bunches of assumptions X and Y, and write X ; Y for the result of not only getting X and Y together but applying the one to the other. Then we may keep the deduction equivalence in the appropriate form

X;A entails B iff X entails $A \rightarrow B$

and continue to use both MPP and CP, as motivated by this equivalence, with semicolons on the left. The rules for introducing and eliminating connectives and quantifiers will be exactly as they were for relevant logic - that is, just Lemmon's rules, minus RAA, with attention paid to the difference between commas and semicolons - but the conventions for matters like counting repetitions of assumptions will be different. For present purposes we do not mind augmentation in the "semicolon" form

X : A leading to X;Y : A

but we do very much object to ignoring repetitions of assumption numbers. So we shall not be able to count X;X as meaning the same as just X. We get

1	(1)	$P \& (P \rightarrow Q)$	A
1	(2)	P	1 &E
1	(3)	$P \rightarrow Q$	1 &E
1;1	(4)	Q	2,3 MPP

for instance, but cannot derive the sequent $P \& (P \rightarrow Q)$: Q. Semantically the idea is to count not just truth and falsehood but degrees to which the truth might be stretched in order to accomodate given vague propositions (e.g. that this patch is red or that that animal is a chook). Then to get P , Q to hold it is necessary to stretch things just far enough to get each of P and Q to hold. To get P ; Q to hold, we have to stretch far enough for P and far enough for Q put together. We can think of "putting together" in this sense as something like adding together. And of course we don't expect X+X to be the same as X.

The philosophical problems raised by the phenomenon of vagueness are extremely difficult and open deep issues both in logic and in the philosophy of language. I cannot pretend to have solved them in these three pages. What I have tried to do is to indicate the nature of the paradox, outline some difficulties facing the various potential solutions and finally sound hopeful about one particular line of response. Most philosophers would disagree with my views both on vagueness and on the prospects for "deviant" logic. Once again, your task is to think about the problem yourself, not to learn to repeat my mistakes.

SHEET 10

In the final part of this course we shall consider one further, and very important, extension of the logical apparatus. This is the addition of a means of describing things as identical or different. "Identical" here is to mean strictly one and the same thing, not just "exactly similar" as it often does in colloquial speech. Examples of true identity statements include:

Tully is Cicero
Hesperus is Venus
George Orwell is Eric Blair
Scotland is Scotland.

Other locutions meaning that Hesperus is Venus include:

Hesperus and Venus are identical
Hesperus is the same thing as Venus
Hesperus and Venus are one and the same.

Do not confuse the "is" of identity with the copula, or "is" of predication. Be clear as to the difference between the roles of "is" in the two sentences:

Edinburgh is no place for the faint-hearted.
Edinburgh is Auld Reekie.

The formal notation for identity is the symbol '='. Where t and u are terms, we form the statement that t is u inserting '=' between them -

$$t = u$$

- as is familiar from primary school arithmetic. Technically, the identity symbol is a dyadic predicate letter, but we write it in infix position for familiarity. It symbolises a relation: the trivial relation which everything has to itself and nothing has to anything else. Negated identity statements will usually be written

$$t \neq u$$

again for reasons of familiarity and convenience.

The rules for introduction and elimination of identity in proofs are quite simple. In the first place, everything is itself, as a matter of logic. Accordingly, the introduction rule for '=' allows any self-identity statement

$$t = t$$

to be introduced on a line by itself, resting on no assumptions at all. The annotation is simply the expression '=I', as there are no line numbers cited. The elimination rule corresponds to another very simple insight into the logical force of the particle. If a and b are one and the same thing then whatever is true of a is true of b (because its being true of a <u>is</u> its being true of b). To put the point another way, if a and b differ in some respect (e.g. a is red and b is not) then they are not identical. This law, or principle, of the "indiscernability of identicals" gives rise to the rule =E:

$$\frac{t = u \qquad A[t]}{A[u]}$$

where A[t] is a formula containing term t, and A[u] results from it by replacing at least one occurrence of t by term u.

It is important that not every occurrence of t in A[t] need be replaced by u (though we allow that it may be). For example:

$a = b \vdash b = a$

1	(1)	a = b	A
	(2)	a = a	=I
1	(3)	b = a	1,2 =E.

Here the first 'a' but not the second in line (2) is replaced by 'b' to get line (3). Note that there are no assumptions for line (2). SI on the above sequent gives us a secondary rule

$$\frac{t = u \qquad A[u]}{A[t]}$$

We shall relax the formalities enough to allow this rule, too, to be called '=E'. So now for =E it does not matter which of the two terms in an identity statement is replaced by the other. This saves some tedium.

The addition of a notation for identity has greatly increased the expressive power of the limited language of first order formal logic. Consider

Ted alone is running	$(x)(Rx \leftrightarrow x = t)$
Everyone except (perhaps) Ted is running	$(x)(x \neq t \rightarrow Rx)$
Ted is the fastest	$(x)(x \neq t \rightarrow Ftx)$
There is more than one runner	$-(x)(y)((Rx \& Ry) \rightarrow x = y)$ or $(\exists x)(\exists y)((Rx \& Ry) \& x \neq y)$

(Rx for 'x is running'. Fxy for 'x is faster than y'.)

The third of these is of some general interest. In English, adjectives which admit of degrees, like "fast", give rise to both comparative and superlative forms. The comparative "faster than" may be formalised as a two-place relation as above. The logic of comparatives can be studied as a subject in its own right. They obey certain laws in virtue of being comparatives:

$$(x)(y)(Fxy \rightarrow -Fyx)$$

$$(x)(y)(z)((Fxy \& Fyz) \rightarrow Fxz)$$

$$(x)(y)((z)(Fxz \leftrightarrow Fyz) \leftrightarrow (z)(Fzx \leftrightarrow Fzy))$$

for instance. Try to see why each of these is true of "faster than". Now the superlative "fastest" is systematically related to the comparative. The fastest is the one faster than any other. To express this relationship we need to be able to say "other", and for this we may use '≠'. Thus the identity symbol yields a reduction of the theory (or at least the logic) of superlatives to that of comparatives which in turn can be represented in the calculus of relations. The effect is explanatory and a conceptual simplification.

The fourth of the above examples also generalises in interesting ways. With identity we have a means of saying there are <u>at least two</u> things of a given kind. Without identity we cannot even say there are two or more things in existence. Note that

$$(\exists x)(\exists y)(Fx \& Fy)$$

does not assert the existence of two Fs, for they must be <u>distinct</u>:

$$(\exists x)(\exists y)((Fx \& Fy) \& x \neq y).$$

An equivalent formula which is slightly shorter is

$$(x)(\exists y)(Fy \& x \neq y).$$

In a similar way we can say that there are at least <u>three</u> Fs:

$$(x)(y)(\exists z)(Fz \;\&\; -(x = z \;\vee\; y = z)).$$

Generally, to say that there are at least n Fs:

$$(x_1)......(x_n)\;(\exists y)\;(Fy \;\&\; -(x_1 = y \;\vee\; .... \;\vee\; x_n = y)).$$

To assert that there are <u>at most</u> two Fs is to deny that there are three or more The same applies to "at most n" for any chosen number n. Hence "at most" is expressible using identity. For example, Unitarianism - "There is one God at most" - is straightforwardly formalisable

$$(\exists x)(y)(Gy \rightarrow x = y)$$

or equivalently

$$(x)(y)((Gx \;\&\; Gy) \rightarrow x = y) \quad \text{or} \quad -(\exists x)(\exists y)((Gx \;\&\; Gy) \;\&\; x \neq y).$$

For "There are at most two Fs" we can use

$$(\exists x)(\exists y)(z)(Fz \rightarrow (x = z \;\vee\; y = z))$$

and generally for any given number n we can express "There are at most n Fs"

$$(\exists x_1).......(\exists x_n)\;(y)\;(Fy \rightarrow (x_1 = y \;\vee\; .... \;\vee\; x_n = y)).$$

Finally, there are <u>exactly</u> three Fs iff there are <u>at least</u> three and <u>at most</u> three. Similarly there are exactly n iff there are at least n and no more than n, whatever number n might be. Thus we can express these "numerically definite quantifiers" by conjoining the appropriate pairs of numerically indefinite ones. In the cases of small numbers, there are neater ways of expressing them using biconditionals:

There is exactly one F	$(\exists x)(y)(Fy \leftrightarrow x = y)$
There are exactly two Fs	$(\exists x)(\exists y)(z)(Fz \leftrightarrow (x = z \;\vee\; y = z))$.

The most important numerically definite quantifier is 'There is exactly one....' which expresses the existence and uniqueness of something satisfying a description. To say there is exactly one F is to assert the existence of such a thing as <u>the</u> F. 'The' is the most commonly used word in written English, and for that reason if for no others its formalisation is of great importance. The doctrine that at least a central class of its uses can indeed be captured in predicate logic with identity is due to Russell and is known as the <u>theory of definite descriptions</u>. The theory has philosophical ramifications and raises problems which are not the concern of this course. The debate as to its correctness or otherwise is still not concluded.

A definite description may be defined roughly as a phrase of the form 'the F'. Such phrases are grammatically rather like proper names. At first sight, they seem to mean much the same as proper names too. It appears a stylistic matter whether one refers to the current head of government as 'Margaret Thatcher' or 'The [first] British prime minister of 1987', and logic is indifferent to style. Consider, however, the phrase 'The President of Scotland in 1987'. This is clearly meaningful despite the fact that there is no such individual. In logic, names are guaranteed to refer to exactly one individual, whereas definite descriptions are not. Moreover, proper names have to be assigned to their bearers by an <u>act</u> of naming whereas a definite description has an internal structure which enables us to understand it, find out which thing it picks out, investigate claims made with it and the like without our having been made party to a specific convention concerning its reference.

Bertrand Russell's idea ('On Denoting', Mind, 1905) for explaining these facts is to construe a sentence like

The Earth's natural satellite is airless

as being really a conjunction. It asserts, says Russell, two things:

(a) The Earth has exactly one natural satellite.

(b) Whatever is a natural satellite of the Earth is airless.

These two, and hence their conjunction, can be expressed in the notation of first order logic with identity, providing a solution to the problem of accounting for the logical behaviour of 'the'. Most neatly:

The F is G $(\exists x)((y)(Fy \leftrightarrow x = y) \ \& \ Gx)$.

Notice that the theory does not provide a direct translation of the definite descriptive phrase 'the F', for the above formula contains no part which can be so read. Such phrases are analysed only in context. That is, we are given not a direct equivalent of the definite description but a way of finding, for any sentence in which such a description occurs, an equivalent one in which no definite description occurs. Such a means of eliminating definite descriptions as linguistic primitives is a contextual definition of them.

Russell's theory of definite descriptions also offers solutions to two more motivating problems. It accounts for definite descriptions which fail to refer by analysing sentences in which they occur as making false claims.

The President of Scotland in 1987 is 6 feet tall

is analysed as a conjunction

There is exactly one president of Scotland in 1987
and any president of Scotland in 1987 is 6 feet tall

which is false because its first conjunct is false. One way for the sample sentence to be false would be for the President to be some other height; but it can turn out false for the alternative reason that there is no president. The Russellian theory allows is to say all this (to make these distinctions, etc.) and to represent its logic within the ordinary first order system.

Secondly, the theory accounts well for the way that understanding novel (newly encountered) definite descriptions differs from understanding novel names. Such understanding amounts to understanding the embedded indefinite descriptions ('...is a satellite', etc.) plus knowing how quantifiers and identity work. This outcome of the theory squares well with common sense. The theory might still be wrong, but it is not just silly; and its success in accounting for this much is at least some evidence in its favour.

Lemmon's treatment of identity in in Chapter 4 Section 3 of Beginning Logic. You are advised to read this section, as it goes into certain details omitted above, especially concerning numerical quantifiers.

To Lemmon's suggestions for further reading I would add five titles. The finest book ever written on logic is H.B. Curry's Foundations of Mathematical Logic (1963) which, however, is very hard. S. Haack's Philosophy of Logics and J.D. McCawley's Everything That Linguists Have Always Wanted To Know About Logic..... (1980) are much more readable and well worth a look. The most approachable texts in mathematical logic beyond what we have covered are probably Metalogic by G. Hunter (out of print but in the library) and Computability and Logic by G. Boolos and R. Jeffrey.