Thinking About Mechanizing the Meta-Theory of Session Types

Francisco Ferreira
(joint work with Nobuko Yoshida)

17th Dec
ABCD Meeting - Imperial College London

Engineering the Meta-Theory of Session Types

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"The limits of my language mean the limits of my world."

–Ludwig Wittgenstein

Who Am I?

- I did my **PhD** at **McGill University**, advised by **Brigitte Pientka.**
- I worked with **Higher Order Abstract Syntax.**
- Also on the **meta-theory** of programming languages.

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  - Beluga — My supervisor's project on **computational reasoning about LF definitions**.
  - Babybel — Our project on supporting **HOAS in functional programming languages** (e.g.: OCaml).
  - Orca — Our project on combining **HOAS and Type Theory.**

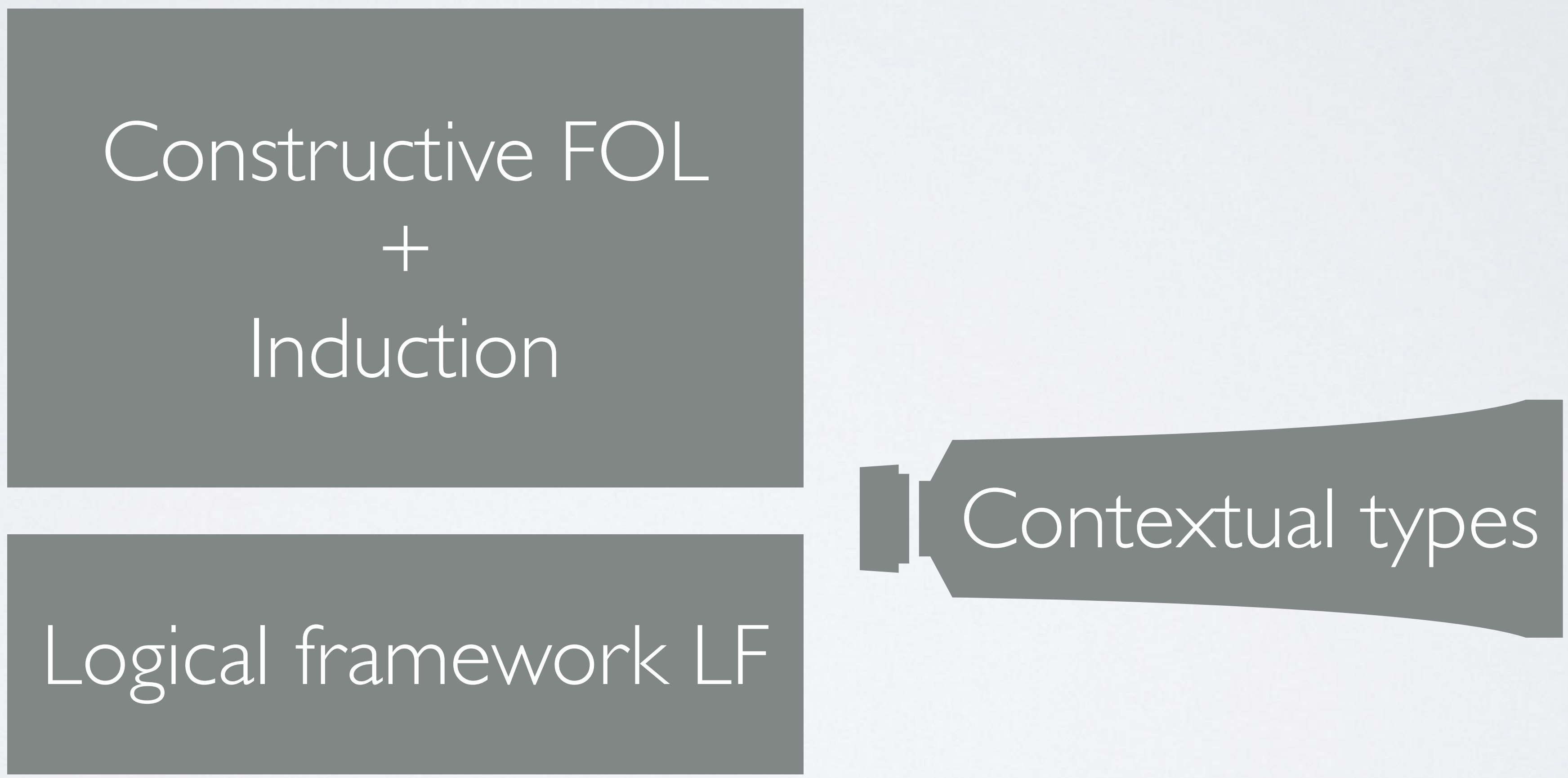
Mechanising the Meta-Theory Session Types

- Names are ubiquitous.
- The binding structure is **quite** rich.
- Channels are handled linearly.
- Names exist besides binders. Names are a first class notion.

The First Step

- Do a case study:
  - Language Primitives and Type Discipline for Structured Communication-Based Programming Revisited, by Yoshida and Vasconcelos.

How Best To Represent Session Types Calculi?



How Best To Represent Session Types Calculi?

Constructive FOL
+
Induction

Nominal Equation Logic

But, Really? Another Proof Assistant?

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- What if we relax the requirement for α-conversion?

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- What if we relax the requirement for α-conversion?
- Work by Ernesto Copello, Maribel Fernandez, et al.
  - Defines a notion of α-compatible relations.
  - Defines a notion of α-structural induction.

But, Really? Another Proof Assistant?

- What if we relax the requirement for α-conversion?

- Work by Ernes Fernandez, et al.

It can be readily implemented in **Agda** and **Coq!**

  - Defines a notion of α-compatible relations.

  - Defines a notion of α-structural induction.

But, Really? Another Proof Assistant?

- What if we relax the requirement for **α**-conversion?
- Work by Erne… …z, et al.
  - Defines a notion of **α**…
  - Defines a notion of **α**-structu… induc…ion.

It can be reac… implement… and **c**…

Induction on judgments is still an "it should be possible" problem in this approach.

Time To Consider Existing Solutions

- Well established work on **Locally Nameless**:
  - Use names for free variables.
  - Use indices for bound variables.
  - Mediate between them with open & close operations.

STLC

$$t \quad := \quad \mathsf{bvar}\, x \quad | \quad \mathsf{fvar}\, p \quad | \quad \mathsf{abs} \quad t \quad | \quad \mathsf{app}\, t\, t$$

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$$\frac{\text{ok } E \qquad (x : T) \in E}{E \vdash \mathsf{fvar}\, x : T} \text{ TYPING-VAR} \qquad \frac{E \vdash t_1 : T_1 \to T_2 \qquad E \vdash t_2 : T_1}{E \vdash \mathsf{app}\, t_1\, t_2 : T_2} \text{ TYPING-APP}$$

$$\frac{\forall x \notin L, \quad E, x : T_1 \vdash t^x : T_2}{E \vdash \mathsf{abs}\, t : T_1 \to T_2} \text{ TYPING-ABS}$$

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Open and close should admit several lemmas:

- Opening locally closed terms does not change the term
- Opening and substitution commute
- The interaction of opening and substitutions of variables

$$\frac{\text{ok } E \quad \cdots}{E \vdash \cdots} \text{ TYPING-APP}$$

The Send Receive System and its Cousins the Relaxed and the Revisited System.

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Language Primitives and Type Discipline for Structured Communication-Based Programming Revisited:
Two Systems for Higher-Order Session Communication

Nobuko Yoshida[1]
Imperial College London

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Start developing the infrastructure and eventually move on to MPST

A Tale of Three Systems

- We set out to represent the three systems described in the paper:
  - The Honda, Vasconcelos, Kubo system from ESOP'98
  - Its naïve but ultimately unsound extension
  - Its revised system inspired by Gay and Hole in Acta Informatica

The Send Receive System

$P ::= \texttt{request}\ a(k)\ \texttt{in}\ P$	session request
$\mid \texttt{accept}\ a(k)\ \texttt{in}\ P$	session acceptance
$\mid k![\tilde{e}]; P$	data sending
$\mid k?(\tilde{x})\ \texttt{in}\ P$	data reception
$\mid k \triangleleft l; P$	label selection
$\mid k \triangleright \{l_1 : P_1 \talloblong \cdots \talloblong l_n : P_n\}$	label branching
$\mid \texttt{throw}\ k[k']; P$	channel sending
$\mid \texttt{catch}\ k(k')\ \texttt{in}\ P$	channel reception
$\mid \texttt{if}\ e\ \texttt{then}\ P\ \texttt{else}\ Q$	conditional branch
$\mid P \mid Q$	parallel composition
$\mid \texttt{inact}$	inaction
$\mid (\nu u)P$	name/channel hiding
$\mid \texttt{def}\ D\ \texttt{in}\ P$	recursion
$\mid X[\tilde{e}\tilde{k}]$	process variables
$e ::= c$	constant
$\mid e + e' \mid e - e' \mid e \times e \mid \texttt{not}(e) \mid \ldots$	operators
$D ::= X_1(\tilde{x}_1\tilde{k}_1) = P_1\ \texttt{and} \cdots \texttt{and}\ X_n(\tilde{x}_n\tilde{k}_n) = P_n$	declaration for recursion

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α-Conversion for Free

- The original system depends crucially on names

$$(\texttt{throw}\ k[k']; P_1) \mid (\texttt{catch}\ k(k')\ \texttt{in}\ P_2) \ \rightarrow\ P_1 \mid P_2$$

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This is a bound variable.

- If α-conversion is built in, this rule collapses to:

$$(\texttt{throw}\ k[k']; P_1) \mid (\texttt{catch}\ k(k'')\ \texttt{in}\ P_2) \rightarrow P_1 \mid P_2[k'/k'']$$

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- If α-conversion is built … o:

$$(\texttt{throw}\ k[k'];P_1) \mid (\texttt{catch}\ k(k'')\ \texttt{i}$$

Locally Nameless makes it impossible to express the original system's name handling!

The Typing Judgement

The rule for parallel composition is where the fun begins:

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$$\frac{\Theta; \Gamma \vdash P \triangleright \Delta \qquad \Theta; \Gamma \vdash Q \triangleright \Delta'}{\Theta; \Gamma \vdash P \mid Q \triangleright \Delta \circ \Delta'} (\Delta \asymp \Delta') \qquad [\textsc{Conc}]$$

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Definition 2.4 (Type algebra) *Typings* Δ_0 *and* Δ_1 *are* compatible, *written* $\Delta_0 \asymp \Delta_1$, *if* $\Delta_0(k) = \Delta_1(k)$ *for all* $k \in \mathrm{dom}(\Delta_0) \cap \mathrm{dom}(\Delta_1)$. *When* $\Delta_0 \asymp \Delta_1$, *the* composition of Δ_0 and Δ_1, *written* $\Delta_0 \circ \Delta_1$, *is given as a typing such that* $(\Delta_0 \circ \Delta_1)(k)$ *is (1)* $\perp$, *if* $k \in \mathrm{dom}(\Delta_0) \cap \mathrm{dom}(\Delta_1)$; *(2)* $\Delta_i(k)$, *if* $k \in \mathrm{dom}(\Delta_i) \setminus \mathrm{dom}(\Delta_{i+1 \bmod 2})$ *for* $i \in \{0, 1\}$; *and (3) undefined otherwise.*

Typing Environments

```
Definition tp_env := {finMap atom_ordType → tp}.

(* lift dual to option *)
Definition option_dual (d : option tp) : option tp :=
  match d with
  | None ⇒ None
  | Some T ⇒ Some (dual T)
  end.

(* compatible envs *)
Definition compatible (D1 D2 : tp_env) : bool :=
  all (fun k ⇒ fnd k D1 == option_dual (fnd k D2))
      (filter (fun k ⇒ k \in supp D1) (supp D2)).

(* composition of envs *)
Definition comp (D1 D2 : tp_env) : tp_env :=
  let: (D1, D12, D2) := split D1 D2 in
  fcat (fcat D1 (update_all_with bot D12)) D2.
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This together requires implementing our own LN infrastructure. But it allows for names and linearity.

The Revisited System

- Now we distinguish between the endpoints of channels.
- It can be represented with LN-variables and names.

Two Kinds of Atoms

```
(* variables that can be substituted
   for channels and expressions *)
Inductive var :=
| Free of VA.atom (* a variable waiting to be instantiated *)
| Bound of nat (* a bound variable *)
.


(* The variables for channel names,
   bound in restrictions (Never substituted) *)
Inductive nvar :=
| NFree of NA.atom
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Channels and Expressions

```
(* Channels use both *)
Inductive channel :=
| Ch of (nvar * polarity) %type (* a channel with polarity *)
| Var of var
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(* Expressions use only variables *)
Inductive exp : Set :=
| tt| ff| ...
| V of var
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```

Processes

```
(* processes bind variables and channels,
   but they are in channels and expressions*)
Inductive proc : Set :=
| par : proc → proc → proc
| send : channel → exp → proc → proc
| receive : channel → proc → proc

| throw : channel → channel → proc → proc
| catch : channel → proc → proc

| nu_nm : proc → proc (* hides a name *)
| nu_ch : proc → proc (* hides a channel name *)
| ...
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```

Binders are “invisible”

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Binders are “invisible”

But Mechanical Proofs Are..

- Well, very mechanical. We have to be very precise with the theorems.

The typing judgements:

```
Inductive oft_exp (G : sort_env) : exp → sort → Prop :=
...

Inductive oft : sort_env → proc → tp_env → Prop :=
...
```

One of the Substitution Lemmas

Lemma 3.1 (Channel Replacement) *If* $\Theta; \Gamma \vdash P \triangleright \Delta \cdot x : \alpha$, *then* $\Theta; \Gamma \vdash P[\kappa^p/x] \triangleright \Delta \cdot \kappa^p : \alpha$.

Proof. A straightforward induction on the derivation tree for P.

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Becomes:

```
Theorem ChannelReplacement G P x kp D:
  def (subst_env_ch x (ce kp) D) →
  oft G P D → oft G (s[ x ⇝ (ch kp)]p P) (subst_env_ch x (ce kp) D).
Proof.
(* ... *)
```

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(* ... *)
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Coq also demanded to be convinced about substituting expressions and various weakening lemmas

Subject Reduction

Theorem 3.3 (Subject Reduction) *If* $\Theta; \Gamma \vdash P \triangleright \Delta$ *with* Δ *balanced and* $P \rightarrow^* Q$*, then* $\Theta; \Gamma \vdash Q \triangleright \Delta'$ *and* Δ' *balanced.*

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Is straightforward to represent:

```
Theorem SubjectReductionStep G P Q D:
  oft G P D → balanced D → P ⟶ Q → exists D', balanced D' /\ oft G Q D'.
Proof.
```

And Lots of Fun To Prove

```
Lemma SubjectReductionStep' G P Q D D' ka:
  oft G P D → balanced D → P --- ka ---> Q → D ~~~ ka ~~~> D' → oft G Q D'.
(* ... *)

Lemma admissible_label P Q:
  P ⟶ Q → exists ka, P --- ka ---> Q.
(* ... *)

Lemma well_typed_step G P Q D ka:
  oft G P D → P --- ka ---> Q → exists D', D ~~~ ka ~~~> D'.
(* ... *)

Lemma typ_step_preserves_balance D D' ka:
  D ~~~ ka ~~~> D' → balanced D → balanced D'.
(* ... *)
```

And Lots of Fun To Prove

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  D ~~~ ka ~~~> D' → balanced D → balanced D'.
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And Lots of Fun To Prove

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Lemma SubjectReductionStep' G P Q D D' ka:
  oft G P D → balanced D → P --- ka ---> Q → D ~~~ ka ~~~> D' → oft G Q D'.
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Lemma admissible_label P Q:
  P ⟶ Q → exists ka, P --- ka ---> Q.
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Lemma well_typed_step G P Q D ka:
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Finally:

```
Theorem SubjectReduction G P Q D:
  oft G P D → balanced D → P ⟶ Q → exists D', balanced D' /\ oft G Q D'.
Proof.
  move⇒Hp Hb Hs.

  apply admissible_label in Hs.
  destruct Hs.
  have HH := well_typed_step Hp H.
  destruct HH.
  exists x0.
  split.
  apply: typ_step_preserves_balance ; [apply: H0 | apply: Hb].
  apply: SubjectReductionStep' ;
    [apply: Hp | apply: Hb | apply: H | apply: H0].
Qed.
```

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- The definition two systems, the unsound proved with a counter example, and the revised with a proof by induction.
- There are still some lemmas to prove (≈4.5 KLOC so far).
- All using a locally nameless representation
- Some use ssreflect and overloaded-lemmas to simply proofs.
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Thanks for your attention. Questions?